Volumes of arithmetic locally symmetric spaces and Tamagawa numbers

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Let G be a Lie group and let μ_G be a left-invariant Haar measure on G. A discrete subgroup $\Gamma \subset G$ is called a *lattice* if the covolume $\mu_G(\Gamma\backslash G)$ is finite. Since the left Haar measure on G is determined uniquely up to a constant factor, this definition does not depend on the choice of the measure.

If Γ is a lattice in G, we would like to be able to compute its covolume. Of course, this depends on the choice of μ_G. In many cases there is a canonical choice of μ, or we are interested only in comparing volumes of different lattices in the same group G so that the overall normalization is inconsequential. But if we don't want to specify μ_G, we can still think of the covolume as a number which depends on μ_G. That is to say, the covolume of a particular lattice Γ is $\mathbb{R}^*$-equivariant function from the space of left Haar measures on G to the group $\mathbb{R}^*/\{\pm 1\}$. Of course, a left Haar measure on a Lie group is the same thing as a left-invariant top form which corresponds to a top form on the Lie algebra of G. This is nice because the space of top forms on $\mathrm{Lie}(G)$ is something we can get our hands on. For instance, if we have a rational structure on the Lie algebra of G, then we can specify the covolume of Γ uniquely up to rational multiple. Even more importantly, we'll see that if G is isomorphic to the real points of an algebraic group, this lets us relate Haar measures on G to Haar measures on the p-adic points.

Now let H be a compact subgroup of G and let X be the homogeneous space G/H. Since H is compact, the pushforward of the Haar measure μ_G to X is a well-defined left-invariant measure on X, which is still unique up to a constant factor. Often we are more interested in the quotient $\Gamma\backslash X$ than we are in $\Gamma\backslash G$. Of course by the definition of the measure on X, the volume of one is the same as the volume of the other. Also note that while $\Gamma\backslash G$ is always a manifold, the quotient $\Gamma\backslash X$ is in general an orbifold. Therefore, we will find it easier to work with the space $\Gamma\backslash G$.

Example 1. *If $G = SL_2(\mathbb{C})$ and $H = SU(2)$ then the homogeneous space G/H has a natural Riemannian metric of constant negative curvature that is preserved by the action of G. Thus, lattices in G correspond to complete hyperbolic three-orbifolds of finite volume. The Mostow rigidity theorem states that if Γ_1 and Γ_2 are two lattices in $SL_2(\mathbb{C})$ that are isomorphic as abstract groups, then they are conjugate. Therefore the hyperbolic structure, and in particular the hyperbolic volume, are topological invariants of the quotient orbifolds.*

In fact, the Mostow rigidity theorem applies to every simple Lie group except for $SL_2(\mathbb{R})$. It follows that for every simple Lie group except $SL_2(\mathbb{R})$, the geometric structure of $\Gamma\backslash G$ depends only on the abstract group Γ.

Example 2. *Two dimensional hyperbolic space is the quotient $SL_2(\mathbb{R})/SO(2)$. In two dimensions the hyperbolic structure of the quotient by Γ is no longer a topological invariant, but the hyperbolic volume still is. If Γ acts cocompactly and freely on X then this follows from Gauss-Bonnet: using the volume form on $SL_2(\mathbb{R})/SO(2)$ corresponding to the hyperbolic metric, the volume of the quotient is -4π times its Euler characteristic. Since the quotient is a $K(\Gamma, 1)$, the Euler characteristic depends only on the abstract group Γ. In fact the same argument works if Γ acts with fixed points or if the quotient is noncompact but still of finite volume (Theorem (66)).*

Example 3. *The moduli space of principally polarized abelian varieties is the quotient of the homogeneous space $Sp_n(\mathbb{Z})/SU(n)$ by the group $Sp_n(\mathbb{Z})$. It follows from Theorem (44) that this space has finite invariant volume.*

In all these examples it's really the quotient of the homogeneous space that is of interest. Moreover, the homogeneous spaces above are all of the form G/K where G is a semi-simple Lie group and K is a *maximal* compact subgroup. These homogeneous spaces are called the symmetric spaces of non-compact type and their quotients are called the locally symmetric spaces of non-compact type. We briefly explain this terminology.

The symmetric spaces of non-compact type, as defined above, admit unique (up to scaling) G-invariant metrics and hence are naturally Riemannian manifolds. On the other hand, suppose M is a Riemannian manifold with the following property: the covariant derivative of its Riemannian curvature tensor is identically zero. Then one can show that the universal cover $\widetilde{M}$ satisfies

1. The isometry group of $\widetilde{M}$ acts transitively.
2. For any point $x \in \widetilde{M}$, there is an isometry φ of $\widetilde{M}$ such that $\varphi(x) = x$ and the action of φ on the tangent space to $\widetilde{M}$ is multiplication by -1.

The second property, which actually implies the first, is how differential geometers characterize a symmetric space. Such spaces have a very elegant classification, which mirrors the classification of semisimple groups. In particular, every symmetric space of negative sectional curvature is of the form G/K where G is a semisimple Lie group and K is a maximal compact subgroup. Since the negatively curved symmetric spaces are noncompact, they're called the symmetric spaces of noncompact type. Motivated by the study of locally symmetric spaces, we will restrict ourselves to studying semi-simple groups G. Also, the theory of covolumes in semi-simple groups seems to be the most inherently interesting.

We make one very important further assumption: that the lattice Γ is arithmetic. We'll define arithmeticity and discuss it at length in the first half of this paper. This assumption allows us to use tools from number theory to solve the geometric problem of computing volumes.

I think that it's important to emphasize at this point that even though restricting to arithmetic groups lets us use tools from the rigid world of number theory, in fact the finite covolume condition is already very rigid. In fact, most lattices in semisimple Lie groups are arithmetic. The strongest theorem to this effect is the Margulis arithmeticity theorem.

Theorem 4. *[11] Let G be a semi-simple Lie group that is not isogeneous to $SO(1, n) \times K$ or $SU(1, n) \times K$ for any compact group K. Let Γ be a lattice in G that is irreducible in the* *sense that no finite index subgroup of* Γ *can be written as a product of lattices in different factors of* G*. Then* Γ *is arithmetic.*

Note that $SL(2,\mathbb{R})$ is isogeneous to $SO(1,2)$ and $SL(2,\mathbb{C})$ is isogeneous to $SO(1,3)$, so that the arithmeticity theorem does not apply to hyperbolic two- and three-manifolds, the two cases that we will work through in detail.

This paper consists of two parts. In the first part, I define arithmetic groups and give a description (Theorem 31)) of all arithmetic subgroups of a Lie group G. I then refine this description for the groups $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$. In the second part, I return to the question of computing covolumes. Section 2.1 contains a calculation of the volumes of $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ and $Sp_{2n}(\mathbb{Z})\backslash Sp_{2n}(\mathbb{R})$. In Sections 2.3 and 2.4 I give a quick introduction to adeles and Tamagawa numbers and explain how the Weil conjecture on Tamagawa numbers (Theorem (61)) translates into formulas for the covolumes of arithmetic lattices. In the last section, I specialize again to the groups $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ and write this formula down in these cases (Theorem (65)).

1 Arithmetic Groups

The goal of the first part of this paper is to define an arithmetic subgroup of a Lie group G. However, arithmetic groups are most naturally defined as subgroups of linear algebraic groups defined over a number field, so this is where we start. We begin with a discussion of linear algebraic groups. We'll use the boldface $\mathbf{G}$ for linear algerbaic groups to distinguish them from the Lie group G. For this section and the next, the group $\mathbf{G}$ does not need to be semisimple.

Definition 5. *A linear algebraic group over a field* k *of characteristic zero is a group object in the category of affine varieties over* k*. In other words, it is an affine variety* $\mathbf{G}$ *defined over* k *such that there is a point* e *in* $\mathbf{G}$ *and morphisms of varieties* $\mu : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ *and* $\iota : \mathbf{G} \to \mathbf{G}$ *satisfying the group axioms of the identity, multiplication, and inversion.*

The following proposition lets us give a more concrete description of a linear algebraic group.

Proposition 6. *A linear algebraic group* $\mathbf{G}$ *always admits a faithful finite dimensional representation.*

Proof. Let R be the algebra $k[x_1, \dots x_n]/I(G)$ of regular functions on $\mathbf{G}$. The group $\mathbf{G}$ acts faithfully on R on the right by pullback, but unfortunately it is infinite dimensional. But by Lemma (7) below, the subrepresentation generated by the functions x_1 through x_n is finite dimensional. Since these functions separate the points of $\mathbf{G}$, the action of $\mathbf{G}$ is still faithful on this finite dimensional subrepresentation. □

Lemma 7. *Any function in the regular representation of* $\mathbf{G}$ *generates a finite dimensional subrepresentation.*

Proof. The multiplication map $\mu : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ induces a map of rings $\mu^* : R \otimes R \to R$. If x is a function in R, decompose $\mu^* x$ into simple tensors:

$$\mu^*(x) = \sum_{i=1}^{d} y_i \otimes f_i.$$

Then for any $g \in \mathbf{G}$,

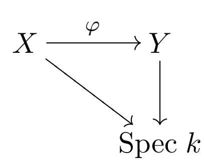
$$g \cdot x(h) = x(hg) = \mu^*(x)(h, g) = \sum_{i=1}^{d} f_i(h) y_i(g).$$

In particular $g \cdot x$ is in the linear span of the d functions f_i. □

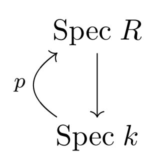
A faithful n-dimensional representation of $\mathbf{G}$ over k gives injective homomorphism defined over k from $\mathbf{G}$ to the general linear group GL_n. The image of any algebraic morphism between affine varieties is an open subset of a Zariski-closed set; if it's also a homomorphism of groups then the image must be Zariski-closed. Therefore Proposition (6) implies that any linear algebraic group is isomorphic to a subgroup of GL_n defined by polynomial equations in the coefficients, for some sufficiently large n.

We end this subsection with a remark on some algebraic geometry preliminaries. Though some theorems are stated more generally, in fact all the varieties that we care about will be affine. The only subtleties will arise from working with affine varieties that are defined over different fields. In order to keep track of this, I find it helpful to use some of the basic notation of schemes. Namely, we wish to identify an affine variety X with its ring of regular functions R; but, in order for arrows to go in the right direction, we will identify it instead with the ringed space Spec R. If this is confusing, reverse the arrows and think about commutative rings.

The variety X is defined over a field k if it comes equipped with a map to Spec k. A morphism $\varphi : X \to Y$ between varieties over k is one that makes the diagram commute:



The most important thing about this perspective is that the variety X is *not* identified with the set of its k-points. Instead, the set of k-points of X is denoted $X(k)$ and is defined to be the set of maps p so that the diagram commutes:



More generally, if A is and commutative k-algebra, the A-points of X, denoted $X(A)$, are the set of maps p so that the following diagram commutes:

$$\begin{array}{ccc} & & \text{Spec } R \\ & \overset{p}{\nearrow} & \downarrow \\ \text{Spec } A & \longrightarrow & \text{Spec } k \end{array}$$

Note that there is a canonical inclusion from the k points of X to the A points of X given by composing $p : \text{Spec } k \to X$ with the map from $\text{Spec } A \to \text{Spec } k$.

Now suppose that $A = L$ is a field. Emphasizing again the distinction between the points of a variety and the variety itself, we define the *extension of scalars* of X from k to L, denoted X_L, as the pullback of X along the map $\text{Spec } L \to \text{Spec } k$. In particular, if A is an extension of L then $X_L(A) = X(A)$. If $X = \text{Spec } R$ is affine, then $X_L = \text{Spec } (R \otimes_k L)$.

Finally, if $X = \mathbf{G}$ is a linear algebraic group over k, then the extension of scalars X_L is a linear algebraic group over L and if A is a k-algebra, the set $X(A)$ inherits the structure of a regular old group.

1.1 Arithmetic Subgroups of Algebraic Groups

Now suppose that k is a number field, i.e. a finite extension of the rational numbers $\mathbb{Q}$, and let $\mathcal{O}_k$ be the ring of integers inside k. Let $GL_n(\mathcal{O}_k)$ be the subgroup of the group $GL_n(k)$ whose matrix entries all lie in the ring $\mathcal{O}_k$. The group $GL_n(\mathcal{O}_k)$ is an example of an arithmetic group.

The notation $GL_n(\mathcal{O}_k)$ looks like the $\mathcal{O}_k$ points of GL_n. If we think of GL_n as a linear algebraic group over a field (for instance $\mathbb{Q}$ would be an obvious choice), then this doesn't work since there are no maps from $\text{Spec } \mathcal{O}_k$ to $\text{Spec } \mathbb{Q}$. But if we want to interpret it as a set of $\mathcal{O}_k$ points, we can. Let M_n be the algebra of $n \times n$ matrices. The linear algebraic group GL_n can be described as the subvariety of $M_n \times \mathbb{A}^1$ defined by the equation $\det(A) \cdot \lambda = 1$. Since all the coefficients of this polynomial are integers, we can think of GL_n as a 'group scheme' over the ring $\mathbb{Z}$. The group $GL_n(\mathcal{O}_k)$ is the group of $\mathcal{O}_k$-points of the group scheme GL_n over $\mathbb{Z}$.

A general arithmetic group is not the $\mathcal{O}_k$ points of a group scheme. But if it is, then you can do something really neat, which is extend scalars to the fields $\mathcal{O}_k/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal. This turns out to be extremely important to the theory of volumes. But when it comes time to do this, we'll do it concretely, so this is the last you'll hear about group schemes.

Let $\mathbf{G}$ be any algebraic group over a number field k. By Proposition (6), there is an injective k-homomorphism from $\mathbf{G}$ to GL_n for some n. Then the intersection of the image of $\mathbf{G}(k)$ with $GL_n(\mathcal{O}_k)$ gives an arithmetic subgroup of $\mathbf{G}(k)$, which we write as $\mathbf{G}(\mathcal{O}_k)$.

More invariantly, let V be a k-vector space on which $\mathbf{G}$ acts faithfully, at least up to isogeny. The natural generalization of $\mathcal{O}_k^n$ to an arbitrary k-vector space V is unfortunately called a lattice, but this should not cause confusion with lattices in Lie groups.

Definition 8. *Let k be a number field. An $\mathcal{O}_k$-lattice (or just a lattice) in a k-vector space V is a finitely generated $\mathcal{O}_k$-submodule M such that $M \otimes_{\mathcal{O}_k} k = V$.*

Remark 9. *If k is a number field, then $k \otimes_{\mathbb{Q}} \mathbb{R}$ is a real vector space of dimension $[k : \mathbb{Q}]$ into which k embeds diagonally (Theorem (26)). It's a basic fact in number theory, and also a trivial consequence of theorem (44) below, that the image of $\mathcal{O}_k$ in $k \otimes_{\mathbb{Q}} \mathbb{R}$ has finite covolume, and so it is also a Lie group lattice. The same is true for any $\mathcal{O}_k$-lattice M in a k-vector space V.*

With M and V as above, let $\mathbf{G}^M(k)$ be the subgroup of $\mathbf{G}(k)$ which fixes the lattice M as a set. We would like our definition of an arithmetic subgroup of $\mathbf{G}$ to include $\mathbf{G}^M(k)$ for any lattice M in any faithful representation V. We give some examples to illustrate how the group $\mathbf{G}^M(k)$ can depend on the choice of M.

Example 10. *If $V = k^n$ and $M = \mathcal{O}_k^n$ then $\mathbf{G}^M(k)$ is exactly $\mathbf{G}(\mathcal{O}_k)$.*

Example 11. *Consider the representation $f : SL_n \to GL_{2n}$ given by*

$$f(g) = A \begin{bmatrix} g & 0 \\ 0 & E_n \end{bmatrix} A^{-1} \qquad A = \begin{bmatrix} E_n & E_n \\ 0 & dE_n \end{bmatrix}$$

where E_n denotes the n-by-n identity matrix. Mutliplying this out gives

$$f(g) = \begin{bmatrix} dg & (E_n - g)/d \\ 0 & E_n \end{bmatrix}$$

so it follows that $f(g) \in GL_{2n}(\mathbb{Z})$ if and only if $g \cong E \mod d$. The corresponding arithmetic subgroup of $SL_n(\mathbb{Z})$ is called the congruence subgroup at level d and is denoted $SL_n(\mathbb{Z}, d)$.

In fact, no matter what faithful representation V and lattice M we pick, the resulting subgroups $\mathbf{G}^M(k)$ are in a sense not too different.

Definition 12. *Let G be any group and let Γ_1 and Γ_2 be two subgroups of G. We say that Γ_1 and Γ_2 are* commensurable *if the intersection $\Gamma_1 \cap \Gamma_2$ has finite index in each.*

Observe that commensurability is an equivalence relation on subgroups of G. We will show that all possible $\mathbf{G}^M(k)$ inside a given group $\mathbf{G}(k)$ are commensurable. The following lemma implies that this is true when $\mathbf{G}$ is a vector space.

Lemma 13. *If M and M' are $\mathcal{O}_k$-lattices in a vector space V over a number field k, then for some $d \in \mathcal{O}_k$ we have $dM \subset M'$.*

Proof. Let $m_1, \dots, m_k$ be a set of generators for M over $\mathcal{O}_k$ and let $m'_1, \dots, m'_l$ be a set of generators for M' over $\mathcal{O}_k$. Since the set $\{m'_1, \dots, m'_l\}$ spans V over k, there are fractions $a^i_j / b^i_j \in k$ with numerator and denominator in $\mathcal{O}_k$ such that

$$\sum_{i=1}^{l} \frac{a^i_j}{b^i_j} m'_i = m_j \qquad \forall j = 1, \dots, k.$$

If $d \in \mathcal{O}_k$ is the least common multiple of all the b^i_j, then it follows that $dM \subset M'$. □

Now we can use this lemma to prove commensurability in any group $\mathbf{G}(k)$.

Proposition 14. *If V and V' are two representations of $\mathbf{G}$ and $M \subset V$ and $M' \subset V'$ are $\mathcal{O}_k$-lattices, then the arithmetic groups $\mathbf{G}^M(k)$ and $\mathbf{G}^{M'}(k)$ are commensurable.*

Proof. (Following [13]) First we prove the proposition when the representations are the same. Since $\mathbf{G}^{M'}(k) = \mathbf{G}^{dM'}(k)$ for any integer d, we can assume that $M' \subset M$ (using the lemma). Apply the lemma again to choose a new integer d so that $dM \subset M'$. Now look at the orbit of M' under $\mathbf{G}^M(k)$ in the space of lattices in V. Since $\mathbf{G}^M(k)$ preserves both M and dM, it follows that for every $g \in \mathbf{G}^M(k)$,

$$dM \subset gM' \subset M.$$

The lattices which lie between M and dM correspond to subgroups of the finite abelian group M/dM; in particular, there are only finitely many. Therefore, the stabilizer of M' in $\mathbf{G}^M(k)$, which is just the intersection $\mathbf{G}^M(k) \cap \mathbf{G}^{M'}(k)$, has finite index in $\mathbf{G}^M(k)$. Reversing the roles of M and M', we conclude that it also has finite index in $\mathbf{G}^{M'}(k)$, and so the two groups are commensurable.

Second, suppose V is not necessarily equal to V'. Using the last paragraph and the fact that commensurability is an equivalence relation, we can replace the lattices M and M' with lattices that have bases over $\mathcal{O}_k$. Therefore, we may assume that $V = k^n, M = \mathcal{O}_k^n$ and $V' = k^m, M' = \mathcal{O}_k^m$. Then the two representations of $\mathbf{G}$ are given by injective morphisms

$$\varphi : \mathbf{G} \to GL_n(k) \quad \text{and} \quad \varphi' : \mathbf{G} \to GL_m(k)$$

and the corresponding arithmetic subgroups are given by

$$G^M(k) = \varphi(\mathbf{G}) \cap GL_n(\mathcal{O}_k) \quad \text{and} \quad \mathbf{G}^{M'}(k) = \varphi'(\mathbf{G}) \cap GL_m(\mathcal{O}_k).$$

Let $\alpha = \varphi' \circ \varphi^{-1}$ be the isomorphism from $\varphi(\mathbf{G})$ to $\varphi'(\mathbf{G})$ as in Figure (1). Since the algebra $k[y_{11}, \ldots, y_{mm}]$ is free, the morphism $\alpha : \varphi(\mathbf{G}) \to M_m(k)$ lifts to a morphism of varieties $\widetilde{\alpha} : M_n(k) \to M_m(k)$.

$$\begin{array}{ccc} M_n(k) & \xrightarrow{\widetilde{\alpha}} & M_m(k) \\ \uparrow & & \uparrow \\ \varphi(\mathbf{G}) & \xrightarrow{\alpha} & \varphi'(\mathbf{G}) \end{array} \tag{1}$$

Of course, $\widetilde{\alpha}$ is not necessarily a homomorphism of algebras. But it has the advantage that we can rescale it. As in proof of the lemma, choose $d \in \mathcal{O}_k$ such that all of the coefficients of the polynomial map $d\widetilde{\alpha}$ are integral. Now define $GL_n(\mathcal{O}_k, d)$ to be the kernel of the reduction-mod-d homomorphism

$$GL_n(\mathcal{O}_k) \to GL_n(\mathcal{O}_k/d\mathcal{O}_k).$$

This is a generalization of the congruence subgroups introduced in example (11). I claim that $\widetilde{\alpha}(GL_n(\mathcal{O}_k, d))$ is contained in $GL_m(\mathcal{O}_k)$.

Let E_n and E_m denote the identity matrices in $GL_n(k)$ and $GL_m(k)$ respectively. Since $\widetilde{\alpha}$ restricts to α, which is a homomorphism of groups, $\widetilde{\alpha}(E_n) = E_m$. Now let's shift $\widetilde{\alpha}$ by the identity matrices. Namely, define $\beta : M_n(k) \to M_m(k)$ by

$$\beta(A) = \alpha(A + E_n) - E_m \quad A \in M_n(k)$$

Then $\beta(0) = 0$, and the coefficients of $d\beta$ are still integral. It follows that

$$\beta(d \cdot M_n(\mathcal{O}_k)) \subset M_m(\mathcal{O}_k).$$

In terms of $\widetilde{\alpha}$, this says that if A is congruent to the identity mod d, then $\widetilde{\alpha}(A)$ is integral. Therefore, $\widetilde{\alpha}(GL_n(\mathcal{O}_k, d))$ is contained in $GL_m(\mathcal{O}_k)$.

Since the mod-d group $GL_n(\mathcal{O}_k/d\mathcal{O}_k)$ is finite, the congruence subgroup $GL_n(\mathcal{O}_k, d)$ has finite index in $GL_n(\mathcal{O}_k)$. Taking the intersection of each with $\varphi(\mathbf{G})$ gives that $\varphi(\mathbf{G}) \cap GL_n(\mathcal{O}_k, d)$ has finite index in $\varphi(\mathbf{G}^M(k))$. By the claim above, α maps this finite index subgroup into $\varphi'(\mathbf{G}^{M'}(k))$. Repeating the above argument with α^{-1} gives a subgroup $\varphi'(\mathbf{G}) \cap GL_m(\mathcal{O}_k, d')$ with finite index in $\varphi'(\mathbf{G}^{M'}(k))$ such that

$$\alpha^{-1}(\mathbf{G}' \cap GL_m(\mathcal{O}_k, d')) \subset \varphi(\mathbf{G}^M(k)).$$

We conclude that the intersection $\mathbf{G}^M(k) \cap \mathbf{G}^{M'}(k)$ has finite index in each, in other words they are commensurable. □

This motivates the definition of an arithmetic subgroup of an algebraic group.

Definition 15. *If $\mathbf{G}/k$ is an algebraic group over a number field k, then an arithemetic subgroup of $\mathbf{G}(k)$ is a subgroup commensurable with $\mathbf{G}^M(k)$ for any faithful representation V containing a lattice M.*

It is natural to ask whether every arithmetic subgroup of $\mathbf{G}$ is the stabilizer of some lattice M. The answer is no, and examples are furnished by negative solutions to the congruence subgroup problem. If we fix a reference representation $\mathbf{G} \to GL_n$, then we can define the congruence subgroups of $\mathbf{G}$ to be the arithmetic subgroups $\mathbf{G} \cap GL_n(\mathcal{O}_k, d)$. It follows from the proof of proposition (14) that any subgroup of $\mathbf{G}$ which is the stabilizer of any lattice in any representation must contain one of these congruence subgroups. However, there are examples of subgroups of finite index in some linear algebraic groups which do not contain any congruence subgroup. Fricke and Klein constructed such a subgroup of $SL_2(\mathbb{R})$ in 1890 as the kernel of a homomorphism from $SL_2(\mathbb{Z})$ to the alternating group A_5 ([4], see [14] for details).

1.2 Arithmetic Subgroups of Lie groups

A linear algebraic group over the real numbers naturally has the structure of a Lie group. Conversely, by studying representations of simple Lie algebras one can show that if G is a connected semi-simple Lie group then G is isogeneous to a group of the form $\mathbf{G}(\mathbb{R})$ where $\mathbf{G}$ is a linear algebraic group defined over $\mathbb{R}$. (Of course, some Lie groups are more naturally though of as complex groups, but we'll see that we can think of these also as linear algebraic groups over $\mathbb{R}$ - see Example (23).) The symmetric space G/K corresponding to G is blind to isogeny of G. Since our professed interest is in symmetric spaces, we lose nothing in imagining that all our Lie groups are in fact the real points of semisimple linear algebraic groups defined over $\mathbb{R}$.

In fact, we can do better. It's not hard to show that every semisimple group over $\mathbb{R}$ can actually be defined over $\mathbb{Q}$. In other words if $\mathbf{G}$ is a semisimple group defined over $\mathbb{R}$, then there is an algebraic group $\mathbf{G}'$ defined over $\mathbb{Q}$ such that the extension of scalars $\mathbf{G}'_{\mathbb{R}}$ is isomorphic to $\mathbf{G}$. In general, we define:

Definition 16. *If L/k is an extension of fields and $\mathbf{G}$ and $\mathbf{G}'$ are groups defined over L and k respectively such that $\mathbf{G}'_L \cong \mathbf{G}$, then we call $\mathbf{G}'$ a k-*form *of $\mathbf{G}$.*

It is important that every semisimple group over $\mathbb{R}$ admits a $\mathbb{Q}$-form because this is what allows us to translate our definition from the last section of arithmetic subgroups of a linear algebraic group over a number field into a definition for Lie groups. Essentially, Γ should be called an arithmetic subgroup of a Lie group G if there is $\mathbb{Q}$-form $\mathbf{G}$ of G, an arithmetic subgroup $\mathbf{G}(\mathbb{Z}) \subset \mathbf{G}$, and an isogeny of Lie groups $\varphi : G \to \mathbf{G}(\mathbb{R})$ such that $\varphi^{-1}(G(\mathbb{Z}))$ is commensurable with Γ. (When we write $\mathbf{G}(\mathbb{Z})$, some representation of G on $\mathbb{Q}^n$ is implicit). But before we make an official definition, we would still like to enlarge the class of arithmetic subgroups in one more way. If G is a product $H \times K$ with K compact and Γ is an arithmetic subgroup of G, then the projection of Γ to H should also be considered an arithmetic subgroup of H. From the point of view of volumes this is completely natural, and similar to the idea that there isn't much difference between volumes of quotients of groups and quotients of symmetric spaces. If $\pi : G \to H$ denotes the projection, then $\Gamma\backslash G$ has finite volume if and only if $\Gamma\backslash H$ does, and if we take normalize the Haar measure on H to be the pushforward of the Haar measure on G then the volumes are the same.

With this in mind, we make the following definition.

Definition 17. *(cf. [12], Dfn. 5.16) A subgroup Γ of a Lie group G is arithmetic if there exists a linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ with an arithmetic subgroup $\mathbf{G}(\mathbb{Z})$, compact normal subgroups K and K' of G and $\mathbf{G}(\mathbb{R})$ respectively, and an isomorphism*

$$\varphi : G/K \to \mathbf{G}(\mathbb{R})/K'$$

such that $\varphi(\overline{\Gamma})$ is commensurable with $\overline{\mathbf{G}(\mathbb{Z})}$, where $\overline{\Gamma}$ and $\overline{\mathbf{G}(\mathbb{Z})}$ are the images of Γ and $\mathbf{G}(\mathbb{Z})$ in G/K and $\mathbf{G}(\mathbb{R})/K$ respectively.

Our Lie groups G will always be semisimple without compact factors. In that case, we can leave out the subgroup K as long as we only require that φ be an isogeny. We now give an example of a non-standard $\mathbb{Q}$-forms of $SL_2(\mathbb{R})$.

Example 18. *Define the quaternion algebra*

$$D_{2,3}(\mathbb{Q}) = \left\{ w + xi + yj + zk \mid \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \in \mathbb{Q}^4 \quad , \quad \begin{array}{c} i^2 = 2 \\ j^2 = 3 \\ k = ij = -ji \end{array} \right\}.$$

For $g = w + xi + yj + zk \in D_{2,3}(\mathbb{Q})$, define $\bar{g} = w - xi - yj - zk$ and $N(g) = g\bar{g}$. Then set

$$SL(1, D_{2,3})(\mathbb{Q}) := \{g \in D_{2,3}(\mathbb{Q}) \mid N(g) = 1\}.$$

Remark 19. *We can think of $D_{2,3}(\mathbb{Q})$ as the $\mathbb{Q}$-points of an algebraic object $D_{2,3}$ defined over $\mathbb{Q}$. In other words, the addition and multiplication on $D_{2,3}(\mathbb{Q})$ comes from morphisms in the category of affine varieties over $\mathbb{Q}$ just like for algebraic groups. This lets us talk about the A-points for any $\mathbb{Q}$-algebra A of $D_{2,3}$ and extend scalars to any field containing $\mathbb{Q}$.*

Similarly $SL(1, D_{2,3})$ is a linear algebraic group defined over $\mathbb{Q}$.

The representation of $D_{2,3}$ on itself gives a morphism to M_4 defined over $\mathbb{Q}$. With respect to the basis $1, i, j, k$, an element $g = w + xi + yj + zk$ acts by the matrix

$$\begin{bmatrix} w & 2x & 3y & -6z \\ x & w & 3z & -3y \\ y & -2z & w & 2x \\ z & -y & x & w \end{bmatrix}. \quad (2)$$

First we show that the $SL(1, D_{2,3})$ is isomorphic to SL_2 when extended to $\mathbb{R}$. Define the $\mathbb{R}$-linear map $\varphi : D_{a,b}(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$\varphi(1) = Id, \quad \varphi(i) = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}, \quad \varphi(j) = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}, \quad \varphi(k) = \begin{bmatrix} 0 & \sqrt{2} \\ -3\sqrt{2} & 0 \end{bmatrix} \quad (3)$$

It can be checked that this is an isomorphism of algebras, and moreover φ takes the norm on $D_{a,b}(\mathbb{R})$ to the determinant on $M_2(\mathbb{R})$, i.e. $\det(\varphi(g)) = w^2 - 2x^2 - 3y^2 + 6z^2$. It follows that φ restricts to an isomorphism between $SL_1(D_{2,3})(\mathbb{R})$ and $SL_2(\mathbb{R})$.

On the other hand, they are not isomorphic over $\mathbb{Q}$. We will prove this by studying the unipotent elements of their $\mathbb{Q}$-points. A unipotent element a of a ring R is an element such that $(a-1)^n = 0$ for large enough n. If $\mathbf{G}$ is a linear algebraic group over k acting on a vector space V, we say that an element $g \in \mathbf{G}(k)$ is unipotent if it is unipotent as an element of the ring of endomorphisms of V. This is equivalent to saying that its characteristic polynomial is $(\lambda - 1)^{\dim(V)}$. A priori this definition depends on the representation of the group, but in fact it is an intrinsic property of the group element g.

Proposition 20. *If $\varphi_1 : \mathbf{G}(k) \to GL_n(k)$ and $\varphi_2 : \mathbf{G}(k) \to GL_m(k)$ are two finite dimensional representations of a linear algebraic group G and g is an element of $\mathbf{G}$ which is unipotent with respect to φ_1, then it is also unipotent with respect to φ_2.*

The proof is elementary (see [Borel, Lin. alg. gps], section I.4.5). It involves characterizing unipotent elements of $\mathbf{G}(k)$ in terms of the regular representation of $\mathbf{G}$.

Now we can prove that $SL_2(\mathbb{Q})$ is not isomorphic to $SL(1, D_{2,3})(\mathbb{Q})$ by the showing that the former has nontrivial unipotent elements while the later does not. The matrices $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ are all unipotent in $SL_2(\mathbb{Q})$. But suppose that $g = w+xi+yj+zk$ were a unipotent element of $SL(1, D_{2,3})(\mathbb{Q})$. We'll show that it can't be unipotent on the representation given by Equation (2) The characteristic polynomial of this matrix is equal to $N(\lambda - g)^2$. Since λ is real and $N(g) = 1$, this simplifies to $(\lambda^2 - 2\lambda w + 1)^2$. Thus g is unipotent if and only if $w = 1$.

However, if $w = 1$ then the equation $N(g) = 1$ reduces to

$$-2x^2 - 3y^2 + 6z^2 = 0$$

A basic divisibility argument shows that this has no rational solutions besides the trivial one. Clearing denominators would then give an integral solution, which we may take to be in lowest terms. But the squares modulo 6 are 0,1,3, and 4, so the equation $2x^2 + 3y^2 = 0$ has no nontrivial solutions mod 6.

This argument also shows that the arithmetic subgroup $SL(1, D_{2,3})(\mathbb{Z}) \subset SL_2(\mathbb{R})$ is not commensurable with $SL_2(\mathbb{Z})$. Indeed, every finite index subgroup of $SL_2(\mathbb{Z})$ contains a unipotent element of the form $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, hence cannot be contained in $SL(1, D_{2,3})(\mathbb{Z})$.

If we're thinking of $SL_2(\mathbb{Z})$ and $SL(1, D_{2,3})(\mathbb{Z})$ as subgroups of the Lie group $SL_2(\mathbb{R})$, then we really want to know more than whether they are commensurable. A better question is whether $SL(1, D_{2,3})(\mathbb{Z})$ is *conjugate* to any subgroup of $SL_2(\mathbb{R})$ which is commensurable with $SL_2(\mathbb{Z})$. If this holds, then we say following [10] that $SL_2(\mathbb{Z})$ and $SL(1, D_{2,3})(\mathbb{Z})$ are *commensurable in the wide sense.* Of course, our unipotent argument shows that these groups are not even commensurable in the wide sense.

1.3 Classification of arithmetic groups

Now that we've seen a few examples of arithmetic groups, we'll try to describe all arithmetic subgroups of a Lie group G. This would involve two steps: first, we would have to find all $\mathbb{Q}$-forms of G as well as all $\mathbb{Q}$-forms of $G \times K$ for (at least some class of) compact groups K; and second, we would have to find every group in each of the resulting commensurability classes. Both steps are really too hard, but we'll see that we can reduce the first step to the more classical question of classifying number fields. As for the second step, a classification of groups within a commensurability class would include a complete solution to the congruence subgroup problem mentioned at the end of section 1.1 , which is known to be very challenging. Therefore, we will focus only on the first step of classifying arithmetic subgroups of $\mathbf{G}$ up to commensurability. If we could do that, then we would at least know all possible covolumes modulo taking rational multiples.

The classification of all $\mathbb{Q}$-forms of a semisimple group $\mathbf{G}$ defined over $\mathbb{R}$ is well understood using Galois cohomology. The complete classification is given in ([12] Fig. 15.2). We will sketch this classification for only the groups $SL_2(\mathbb{C})$ and $SL_2(\mathbb{R})$ in Theorem (33), which will ultimately give a pretty thorough answer for the volume of arithmetic hyperbolic two- and three-manifolds. But before we do, we need to present one more construction of arithmetic groups, through restriction of scalars. It will clarify both the role of the compact factor K in the definition of arithmetic groups and the reason we defined algebraic groups over a general number field k even though the definition of arithmetic groups only involved $\mathbb{Q}$.

Let L/k be an extension of fields of finite degree, and let X be a variety defined over L. The restriction of scalars (also called Weil restriciton) of X from L to k is a variety defined over k and we will denote it $\mathrm{Res}_{L/k}X$. A mnemonic - which isn't far from a definition - is that the points of X and $\mathrm{Res}_{L/k}X$ are the same, where 'points' refers to those points over the field of definition:

$$\mathrm{Res}_{L/k}X(k) \cong X(L).$$

Contrast this with extension of scalars in which, if Y is a variety defined over k, then

$$Y_L(L) = Y(L).$$

We can turn this mnemonic into a definition by requiring it to hold for points over all commutative k-algebras. Namely, if S is a commutative k-algebra, there should be a natural correspondence

$$\text{Res}_{L/k}X(S) \cong X(S \otimes_k L). \tag{4}$$

This is equivalent to saying that restriction scalars is the pushforward in the category of affine schemes, which implies that $\text{Res}_{L/k}X(S)$ is unique if it exists. To show it exists, we construct it explicitly. Suppose $X = \text{Spec } L[x_1, \dots x_n]/(p_1(x_i), \dots, p_r(x_i))$. Let $\alpha_1, \dots, \alpha_d$ be a basis for L over k, where d is the degree of the extension. Write each coordinate x_i and each polynomial p_k in this basis:

$$x_i = \sum_{j=1}^{d} y_i^j \alpha_j \qquad \text{and} \qquad p_k(\{x_i\}) = \sum_{j=1}^{d} q_k^j(\{y_i^j\})\alpha_k$$

Then set $R = k[\{y_i^j\}]/(\{q_k^j\}$ and $Y = \text{Spec } R$. If $\varphi : R \to S$ is an S-point of Y, one checks that

$$\begin{aligned} \widetilde{\varphi} : L[\{x_i\}]/(\{p_i\}) \to & \quad S \otimes_k L \\ x_i \mapsto & \quad \sum_{j=1}^{d} \varphi(y_i^j) \otimes \alpha_j \end{aligned}$$

is an $S \otimes L$ point of X and the map $\varphi \mapsto \widetilde{\varphi}$ gives the correspondence (4).

Since this was confusing to me, let me point out that there is another functor from schemes over L to schemes over k which is *not* restriction of scalars. Namely, if Y is a scheme over L, you could just compose the morphism $Y \to \text{Spec } L$ with the morphism $\text{Spec } L \to \text{Spec } k$. Let's compare these for $Y = \text{Spec } \mathbb{C}[x]$. Then

$$\text{Res}_{\mathbb{C}/\mathbb{R}} Y = \text{Spec } \mathbb{R}[y_1, y_2]$$

but under this other functor, Y is still $\text{Spec } \mathbb{C}[x]$, which we can also write as $\text{Spec } \mathbb{R}[y, x]/(y^2 + 1)$. This is a pretty weird scheme; for instance, it has no points over $\mathbb{R}$.

Perhaps it is also helpful to compare the points of X and $\text{Res}_{L/k}X$ over the same field. This is easiest for the algebraic closure $\overline{k}$. We do it first dually for the fields.

Proposition 21. *If L is a degree d extension of k then the ring $\overline{k} \otimes_k L$ is isomorphic as a ring to $\overline{k}^d$. The map $1 \otimes id$ from L into $\overline{k} \otimes_k L$ sends L to the sum of its Galois embeddings.*

Proof. Let $\alpha_1, \dots, \alpha_d$ be a basis for L over k. Then they are also a basis over $\overline{k}$ for the $\overline{k}$-vector space $\overline{k} \otimes_k L$. More over, each α_i acts by a linear transformation on this vector space. Since $\overline{k}$ is algebraically closed and the linear transformations corresponding to the α_i all commute, they are mutually diagonalizable. Each eigenspace gives an embedding L into $\overline{k}$, i.e. a Galois embedding of L.

From the universal property of tensor product, precomposition with the map $L \to \overline{k} \otimes_k L$ gives a natural transformation between maps $L \to k$ and maps $\overline{k} \otimes_k L \to k$. It follows that each Galois embedding of L corresponds to exactly one eigenspace. □

Now if X is a variety over L and if σ is a Galois embedding of L, then we can pull back X along the morphism σ to define the extension of scalars of X. To emphasize that this depends on σ, we denote the resulting variety over $\overline{k}$ by X^σ.

$$\begin{array}{ccc} X^\sigma & & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \overline{k} & \xrightarrow{\sigma} & \operatorname{Spec} L \end{array}$$

Then Proposition (21) gives a description of the $\overline{k}$-points of $\operatorname{Res}_{L/k}X$.

Corollary 22.

$$\operatorname{Res}_{L/k}X(\overline{k}) = \prod_{\sigma:L\to\overline{k}} X^\sigma(\overline{k}).$$

Proof. The left hand side is $X(\overline{k} \otimes_k L)$. By Proposition (21), Spec $\overline{k} \otimes_k L$ consists of d points, each mapping to Spec L under a different Galois embedding σ. □

As an example, we can clarify the correspondence between semisimple Lie groups and linear algebraic groups defined over $\mathbb{R}$. One might worry that semisimple Lie groups can be either real or complex, but this example shows that we can think of the complex ones as linear algebraic groups defined over $\mathbb{R}$ as well.

Example 23. *If* $\mathbf{G}$ *is a group defined over* $\mathbb{C}$ *then* $\operatorname{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}$ *is a group defined over* $\mathbb{R}$ *whose* $\mathbb{R}$*-points correspond to the* $\mathbb{C}$*-points of* $\mathbf{G}$. *In other words,* $\mathbf{G}(\mathbb{C})$ *and* $(\operatorname{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G})(\mathbb{R})$ *give the same Lie groups. On the other hand,* $(\operatorname{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G})(\mathbb{C})$ *is isomorphic as a Lie group to* $\mathbf{G}(\mathbb{C}) \times \mathbf{G}(\mathbb{C})$. *Note that restricting from* $\mathbb{C}$ *to* $\mathbb{R}$ *multiplies the dimension of the variety by 2, the degree of the extension.*

Our main application of restriction of scalars will be when $k = \mathbb{Q}$, L is a number field, and $X = \mathbf{G}$ is an algebraic group over L. Also, we're not so much interested in the $\overline{\mathbb{Q}}$-points of $\operatorname{Res}_{L/\mathbb{Q}}X$ as we are in the real points and eventually the p-adic points. We will now describe the $\mathbb{R}$-points using the method of descent.

First of all, Proposition (21) clearly remains valid if we replace $\overline{k}$ replaced with any algebraically closed field containing k. In particular, if $k = \mathbb{Q}$, it says that

$$\operatorname{Res}_{L/\mathbb{Q}}\mathbf{G}(\mathbb{C}) = \prod_\sigma \mathbf{G}^\sigma(\mathbb{C})$$

The Galois group of $\mathbb{C}$ over $\mathbb{R}$ acts on the set of $\mathbb{C}$ points of a variety X defined over $\mathbb{R}$ by precomposing the map Spec $\mathbb{C} \to X$ with the automorphism of $\mathbb{C}$. The key is that $X(\mathbb{R})$ is exactly the set of complex points that are invariant under this action. This makes sense if we think of Spec $\mathbb{R}$ as the quotient of Spec $\mathbb{C}$ by the action of the Galois group.

Applying this to $\operatorname{Res}_{L/\mathbb{Q}}\mathbf{G}$, whose $\mathbb{C}$ points are maps

$$p : L \otimes_\mathbb{Q} \mathbb{C} \to \mathbf{G}$$

we see that its $\mathbb{R}$-points correspond to maps from the quotient $(L \otimes_\mathbb{Q} \mathbb{C})/\operatorname{Gal}(\mathbb{C}/\mathbb{R})$. Let's understand this quotient. The Galois group is of course just generated by complex conjugation, and by Proposition (21) we have

$$L \otimes_\mathbb{Q} \mathbb{C} = \bigoplus_\sigma \mathbb{C}.$$

Complex conjugation acts on the second factor on the left hand side, and the fixed subring is exactly $L \otimes_{\mathbb{Q}} \mathbb{R}$. On the right hand side, the action conjugates every copy of $\mathbb{C}$, but it also permutes them. If $\sigma : L \to \mathbb{C}$ has dense image, then the conjugate $\overline{\sigma}$ is a different Galois embedding, but if the image of σ is contained in $\mathbb{R}$ then $\overline{\sigma} = \sigma$. An *infinite place*, or simply a *place*, of L is defined to be an equivalence class of Galois embeddings under the action by conjugation. If $\sigma = \overline{\sigma}$, then we say that σ is a real place and if $\sigma \neq \overline{\sigma}$, we say that the class of σ is a complex place. It's also common to refer to a Galois embedding itself as a complex place instead of the equivalence class. So we might also speak of the set of places of L as a subset of Galois embeddings containing one representative of each conjugate pair.

We conclude that

$$(L \otimes_{\mathbb{Q}} \mathbb{C})/\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{C}^{r_1} \oplus \mathbb{R}^{r_2}$$

where r_1 is the number of complex places of L and r_2 is the number of real places. It follows that

$$\mathrm{Res}_{L/\mathbb{Q}}\mathbf{G}(\mathbb{R}) = \prod_{\mu} \mathbf{G}^{\mu}(\mathbb{C}) \times \prod_{\nu} \mathbf{G}^{\nu}(\mathbb{R})$$

where μ runs over all complex places of L and ν runs over all real places. It's important to notice that although the groups $\mathbf{G}^{\mu}$ are all isomorphic as linear algebraic groups over $\mathbb{C}$, the groups $\mathbf{G}^{\nu}$ are *not* all isomorphic as linear algebraic groups over $\mathbb{R}$. All that we can say is that $\mathbf{G}^{\nu}$ is some $\mathbb{R}$-form of $\mathbf{G}_{\mathbb{C}}$.

Example 24. *Let $L = \mathbb{Q}[\alpha]/(\alpha^2 - 2)$, and let $\mathbf{G}$ be the subgroup of $GL_3(L)$ which preserves the quadratic form*

$$x^2 + y^2 - \alpha z^2.$$

Since L has 2 real places and no complex places, the Lie group $\mathrm{Res}_{L/\mathbb{Q}}\mathbf{G}(\mathbb{R})$ is a product of two real forms of $\mathbf{G}_{\mathbb{C}}$. The two real forms are the special orthogonal groups preserving the quadratic forms

$$x^2 + y^2 - \sqrt{2}z^2 \qquad \text{and} \qquad x^2 + y^2 + \sqrt{2}z^2$$

respectively. Since the first form has signature $(2, 1)$ and the second form is positive definite, the two real forms of $\mathbf{G}_{\mathbb{C}}$ are not isomorphic.

Now let Γ be an arithmetic subgroup of $\mathbf{G}(L)$. Since the L-points of $\mathbf{G}$ and the $\mathbb{Q}$-points of $\mathrm{Res}_{L/\mathbb{Q}}\mathbf{G}$ are naturally isomorphic, we can also think of Γ as a subgroup of $\mathrm{Res}_{L/\mathbb{Q}}\mathbf{G}(\mathbb{Q})$. Fortunately, it is still arithmetic

Proposition 25. *An arithmetic group remains arithmetic under restriction of scalars.*

Proof. Since $\mathbf{G}(L)$ and $\mathrm{Res}_{L/\mathbb{Q}}\mathbf{G}(\mathbb{Q})$ are isomorphic as groups, commensurability in one is the same as commensurability in the other, so it's enough to show that some arithmetic subgroup $\mathbf{G}(\mathcal{O}_L)$ of $\mathbf{G}$ is also an arithmetic subgroup of $\mathrm{Res}_{L/\mathbb{Q}}\mathbf{G}$. We choose $\mathbf{G}(\mathcal{O}_L)$ to be the stabilizer of the lattice $\mathcal{O}_L^n$ in a faithful representation of $\mathbf{G}$ on L^n.

A representation of $\mathbf{G}$ on L^n is a homomorphism from $\mathbf{G}$ to $GL_n(L)$; under restriction of scalars this gives a homomorphism

$$\mathrm{Res}_{L/\mathbb{Q}}\mathbf{G} \to \mathrm{Res}_{L/\mathbb{Q}}GL_n \cong GL_{dn}$$

The subgroup $\mathbf{G}(\mathcal{O}_L)$ is still the stabilizer of $\mathcal{O}_L^n$, so we just need to show that that latter is a $\mathbb{Z}$-lattice in $\mathbb{Q}^{nd}$, where the latter is $\mathrm{Res}_{L/\mathbb{Q}}L$. Clearly it's enough to show $\mathcal{O}_L$ is a $\mathbb{Z}$-lattice in L. Since it is finitely generated as a $\mathbb{Z}$-module, this is equivalent to showing

$$\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Q} = L.$$

Let a/b be any element of k, with a and b in $\mathcal{O}_k$. Using the definition of the ring of integers, b is a root of some irreducible monic polynomial

$$x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0 \qquad a_i \in \mathbb{Z}.$$

Set

$$b' = b^{r-1} + c_{r-1}b^{r-2} + \cdots + c_1.$$

Clearly b' is still an algebraic integer, and $bb' = -c_0 \in \mathbb{Z}$, so

$$\frac{a}{b} = ab' \times \frac{1}{bb'} \in \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Thus we conclude $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Q} = L$ and the proposition is proved. □

Remark 26. *Since $\mathbb{Z}$-lattices in $\mathbb{Q}$-vector spaces have finite covolume when you tensor with $\mathbb{R}$, this shows that an $\mathcal{O}_k$-lattice in a k-vector space V is a Lie group lattice in the real vector space $\mathrm{Res}_{k/\mathbb{Q}}V(\mathbb{R})$ (cf. Remark 9). The* discriminant *of the number field k, denoted Δ_k, is defined to be the square of the covolume of $\mathcal{O}_k$ in $\mathrm{Res}_{k/\mathbb{Q}}k(\mathbb{R})$, where the measure on $\mathbb{C}$ is defined to be $2|dz|^2$.*

Let's summarize the recipe for constructing arithmetic subgroups of a Lie group G using restriction of scalars. First, if G can be written as the product $G_1 \times G_2$ where G_1 and G_2 have different Cartan types, then the construction reduces to a separate construction on G_1 and G_2.

Definition 27. *A semisimple Lie group or linear algebraic group is* isotypic *if all its simple factors are of the same Cartan type. In the case $\mathbf{G}$ is a linear algebraic group, this is the same as saying that each simple factor becomes isomorphic when you extend scalars to $\mathbb{C}$.*

Assume that G is isotypic. The next step is to find a number field L and a simple linear algebraic group $\mathbf{G}$ defined over L so that $\mathrm{Res}_{L/\mathbb{Q}}\mathbf{G}(\mathbb{R})$ is isogenous to the product of G with some number of copies of the compact form of G. Then, take any arithmetic subgroup of the algebraic group $\mathbf{G}$ and project it to G. Conveniently, this construction only involves compact factors that are isomorphic to a number of copies of the compact form of G.

We'll see that this construction produces all arithmetic subgroups of a Lie group $\mathbf{G}$. To be more specific, we first need to define irreducibility.

Definition 28. *Let G be a semisimple Lie group with and Γ an arithmetic subgroup. We say Γ is* reducible *(else,* irreducible*) if there exists a decomposition $G = G_1 \times G_2$, with neither G_1 nor G_2 compact, and arithmetic subgroups $\Gamma_1 \in G_1$ and $\Gamma_2 \in G_2$ such that Γ is commensurable with $\Gamma_1 \times \Gamma_2$.*

Now let Γ be any irreducible arithmetic subgroup of a semisimple Lie group G. For simplicity, we assume that G is simply connected and has no compact factors. Then there exists a semisimple linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, an arithmetic subgroup $\mathbf{G}(\mathbb{Z})$, a compact group K, and an isogeny $\varphi : G \times K \to \mathbf{G}(\mathbb{R})$ so that Γ is commensurable with $\varphi^{-1}(\mathbf{G}(\mathbb{Z}))/K$. We may as well assume that K is minimal in the sense that Γ cannot be constructed using a subgroup or quotient of K. Under this assumption, it follows that the linear algebraic group $\mathbf{G}$ cannot be written as a nontrivial product of two linear algebraic groups defined over $\mathbb{Q}$.

Indeed, suppose $\mathbf{G} \cong \mathbf{G}_1 \times \mathbf{G}_2$. Taking $\mathbb{R}$ points, we would have that $G \times K$ was isogeneous to $\mathbf{G}_1(\mathbb{R}) \times \mathbf{G}_2(\mathbb{R})$. Since Γ is irreducible, the quotient by K cannot split into a product of two noncompact groups, hence either $\mathbf{G}_1$ or $\mathbf{G}_2$ must be compact. Suppose it is $\mathbf{G}_2$. Then $\mathbf{G}_1(\mathbb{R})$ is isogenous to G times a subgroup of K, which contradicts the minimality of K.

The conclusion is that if Γ is irreducible, then it can be constructed using a linear algebraic group $\mathbf{G}$ which is simple over $\mathbb{Q}$, in the sense that it has no connected normal subgroups defined over $\mathbb{Q}$ (or since we can take $\mathbf{G}$ to be semisimple, this is equivalent to saying that it does not split as a product over $\mathbb{Q}$). We stress that this is not the same as saying that the Lie groups $\mathbf{G}(\mathbb{R})$ or $\mathbf{G}(\mathbb{C})$ are simple. In general, we use the following terminology.

Definition 29. *A linear algebraic group* $\mathbf{G}$ *defined over a field* k *is called* absolutely simple *if* $\mathbf{G}_{\overline{k}}$ *is simple.*

The next proposition shows that any simple linear algebraic group over $\mathbb{Q}$ becomes absolutely simple if we think of it over the right field.

Proposition 30. *Let* $\mathbf{G}$ *be a simple algebraic group defined over* $\mathbb{Q}$*. Then there exists a unique number field* k *and an absolutely simple group* $\mathbf{G}'$ *defined over* k *such that* $\mathbf{G} \cong \mathrm{Res}_{k/\mathbb{Q}}\mathbf{G}'$

Proof. The key to the proof is the principle of descent. In general, let L be a Galois extension of a field k, and let X be a variety defined over L. Suppose that the action of $\mathrm{Gal}(L/k)$ on $\mathrm{Spec}\ L$ lifts to an action on X. Then X is isomorphic to the extension of scalars X'_L for some variety X' defined over k. If X is affine, then it's not hard to prove this explicitly; on the other hand, this is just the statement that we can take quotients by finite group actions (or if L is an infinite Galois extension, by profinite group actions).

Since $\mathbf{G}_{\overline{\mathbb{Q}}}$ is a semisimple group over an algebraically closed field, it can be written as a product of absolutely simple factors $\mathbf{G}_1 \times \cdots \times \mathbf{G}_n$. Since $\mathbf{G}$ is defined over $\mathbb{Q}$, the action of $\mathrm{Gal}_{\overline{\mathbb{Q}}/\mathbb{Q}}$ is easily seen to lift to an action on $\mathbf{G}_{\overline{\mathbb{Q}}}$. But this action doesn't need to preserve the decomposition of $\mathbf{G}_{\overline{\mathbb{Q}}}$ into its factors. In fact, the principle of descent implies that it must act transitively on the fibers since each orbit descends to a group defined over $\mathbb{Q}$ and $\mathbf{G}$ is assumed to be simple over $\mathbb{Q}$. Let $H \subset \mathrm{Gal}_{\overline{\mathbb{Q}}/\mathbb{Q}}$ be the stabilizer of $\mathbf{G}_1$, and let k be the extension of $\mathbb{Q}$ fixed by H. We apply the principle of descent again to show that $\mathbf{G}_1$ can be defined over k.

Then I claim $\mathrm{Res}_{k/\mathbb{Q}}\mathbf{G}_1 = \mathbf{G}$. I can show this at the level of $\mathbb{Q}$-points, but I haven't quite worked out why the claim is true yet. □

In particular, this proposition shows that the irreducible arithmetic subgroup Γ can be constructed from restriction of scalars from an absolutely simple group $\mathbf{G}'$. Therefore, to find all irreducible arithmetic subgroups of G, we no longer need to worry at all about the compact factor K; instead, it's enough to classify the k-forms in the Cartan type of G for each number field k. We summarize this conclusion in the following theorem.

Theorem 31. *(cf. [12], Theorem 5.50) If G is a semisimple Lie group with no compact components and Γ is an irreducible arithmetic subgroup in G, then there exist*

1. *an algebraic number field k*
2. *a connected, absolutely simple linear algebraic group $\mathbf{G}$ defined over k with an arithmetic subgroup $\mathbf{G}(\mathcal{O}_k)$, and*
3. *an isogeny $\varphi : \mathrm{Res}_{k/\mathbb{Q}}\mathbf{G}(\mathbb{R}) \to G$*

such that $\varphi(\mathbf{G}(\mathcal{O}_k))$ is commensurable with Γ.

1.4 Classification for SL_2

We now apply Theorem (31) to classify all commensurability classes of arithmetic subgroups of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$. We'll do the real case in detail. According to the theorem, we need to find all $\mathbb{Q}$-forms of SL_2 as well as all k-forms $\mathbf{G}$ of SL_2 for each number field k such that

$$\mathrm{Res}_{k/\mathbb{Q}}\mathbf{G}(\mathbb{R}) \cong SL_2(\mathbb{R}) \times SU(2) \times \cdots \times SU(2)$$

The first observation is that every place of k must be real. We don't want to have to classify totally real number fields, so let's suppose we start with one. Then the next step is to find all k forms of SL_2. We approach this problem by replacing it with the linear problem of classifying all k-forms of the algebra M_2 of two-by-two matrices. These are called the 'central simple algebras' of rank 4.

Definition 32. *A* central simple algebra *over a number field k is a finite dimensional k-algebra B such that the extension $B \otimes_k \overline{k}$ of B to the algebraic closure of k is isomorphic to a matrix algebra.*

If B is a central simple algebra, then we define the norm of an element of B to be the determinant of the corresponding element of the matrix algebra $B \otimes_k \overline{k}$.

Theorem 33. *Let k be an algebraic number field. The set of k-forms of SL_2 is in bijection with the set central simple algebras of rank 4. Given a central simple algebra B, the corresponding k-form of SL_2 is the group of elements of B of norm 1. We use the natural notation $SL_1(B)$ to refer to this group.*

Proof. We will give the essence of the argument while avoiding all the proofs. Progressively more complete versions of this argument are given in Serre's books on Galois cohomology and Local fields ([15] and [16]). The idea is that the k-forms of an algebraic object are determined by the automorphism group of the object over the algebraic closure. Since the automorphism group of SL_2 over k and the automorphism group of M_2 over k are both equal to PGL_2, there should be a correspondence between their k-forms. We can make this more precise using Galois cohomology.

Recall that if G is a discrete group and M is a G-module, then the group cohomology of G with coefficients in M, $H^*(G, M)$, is defined to be the cohomology of the functor taking M to its set of G-invariants. If we construct the classifying space EG of G through the standard simplicial construction, we can view M as a local system on this simplicial complex; from this, we can build in the usual way a chain complex whose cohomology groups are $H^*(G, M)$.

If we replace the module M by a non-abelian group A we can try to mimic this construction. This is kind of tricky in general, but in low dimensions it's not so bad; by choosing appropriate nonabelian versions of the differentials in the abelian complex, we can define the nonabelian cohomology $H^0(G, A)$ and $H^1(G, A)$ as pointed sets. The former is still the set of G-invariants in A (the point is the identity of A). The latter, it turns out, is naturally identified with the pointed set of isomorphism classes of principle homogeneous spaces for A, meaning the set of G-spaces P admitting a simply transitive right A action that is equivariant with respect to G ([15], p. 47). (The point in the set of principle homogeneous spaces is A itself). Note that it's because of the G action that there can be more than one isomorphism class of principle homogeneous space.

The setting we are interested is when G is the absolute Galois group of a Galois extension L/k and the group A is the set of L-points of an algebraic group X defined over k. (More specifically, $X = SL_2$). As we've seen, G acts on $X(L)$ by precomposition. In fact we're really interested in the limit of the Galois group over all extensions L of k, or equivalently in the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. It's fine to just think about the absolute Galois group, but in order to do that we need to equip it with the profinite topology and require all its actions to be continuous.

Now we can formulate more carefully the principle behind the proof of this theorem. Let L be a Galois extension of k, possibly of infinite degree. Let X be an algebraic object defined over k, by which we mean that X is an algebraic variety possibly with extra structure. Let A be the automorphism group of X, so that A is an algebraic group defined over k. Then the principle is that the pointed set of k-isomorphism classes of k-forms of X_L (the point being X) is naturally in bijection with the pointed set $H^1(\mathrm{Gal}(L/k), A(L))$. Though we won't prove this, it's maybe helpful to write down the bijection.

If X' is a k-form of X_L then let $\theta(X')$ denote the set of isomorphisms defined over L from X'_L to X_L. The group $A(L)$ acts simply transitively on the right on $\theta(X')$ by post-composition. Also the Galois group acts on $\theta(X')$ by conjugation: an element γ of the Galois group lifts to an action on X_L and an action on X'_L, and the composition $\gamma \circ \varphi \circ \gamma^{-1}$ descends to the trivial action on L, so it defines another isomorphism over L. Moreover, this Galois action is compatible with the Galois action on $A(L)$. We conclude that $\theta(X')$ is a principle homogeneous space for the group $A(L)$, so it corresponds to an element of $H^1(\mathrm{Gal}(L/k), A)$.

In the case where X is an algebraic group or where X is an algebra over k, Serre [16] proves explicitly that the map θ is a bijection from the set of isomorphism classes of k-forms to $H^1(\mathrm{Gal}(L/k), A(L))$. This establishes the existence of a bijection between k-forms of SL_2 and central simple algebras of rank 4. The fact that the correspondence is by taking the elements of norm 1 can be deduced from the fact that that procedure is sufficiently functorial.

□

We have now reduced the problem to classifying all rank 4 central simple algebras. It turns out that these are all generalizations of the quaternion algebra $D_{2,3}$ that we used to construct a nonstandard $\mathbb{Q}$-from of SL_2 in Example (18).

Definition 34. *Let k be a field of characteristic different from 2, and let a and b be in k^*. The quaternion algebra $D_{a,b}(k)$ is the algebra*

$$D_{a,b}(k) = \left\{ w + xi + yj + zk \mid \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \in k^4 \quad , \quad \begin{matrix} i^2 = a \\ j^2 = b \\ k = ij = -ji \end{matrix} \right\}.$$

As in Remark (19), we think of $D_{a,b}(k)$ as the k-points of an algebraic object $D_{a,b}$ defined over k. We quote the following theorem on central simple algebras.

Theorem 35. *([10], Theorem 2.1.8) Every central simple algebra of rank 4 over a field of characteristic different from 2 is a quaternion algebra.*

Example 36. *The matrix algebra $M_2(k)$ is isomorphic to the quaternion algebra $D_{1,1}(k)$ under the identifications*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mapsto i \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mapsto j.$$

Let's take stock of what we've shown so far. From theorems (this one), (the one about everything coming from restriction of scalars) and the (galois cohomolgy one), we know that every commensurability class of arithmetic subgroups of $SL_2(\mathbb{R})$ can be constructed from a pair $(k, D_{a,b})$ where k is a totally real number field and a and b are elements of k^*. In order to call this a classification, we would also like to know when a pair $(k, D_{a,b})$ actually produces a commensurability class of subgroups and when two pairs produce the same class.

The only condition we need for the pair $(k, D_{a,b})$ to produce an arithmetic subgroup of $SL_2(\mathbb{R})$ is that the semisimple Lie group

$$\mathrm{Res}_{k/\mathbb{Q}}(SL_1(D_{a,b}))(\mathbb{R})$$

have a single noncompact simple factor. Recall that for any group $\mathbf{G}$ defined over k,

$$\mathrm{Res}_{k/\mathbb{Q}}\mathbf{G}(\mathbb{R}) = \prod_{\sigma} \mathbf{G}^{\sigma}(\mathbb{R})$$

where the product is over the infinite places and $\mathbf{G}^{\sigma}$ is the extension of scalars from k to $\mathbb{R}$ or $\mathbb{C}$ along the morphism σ. Since the norm on $D_{a,b}$ doesn't change under extension of scalars, $SL_1(D_{a,b})^{\sigma}(\mathbb{R})$ is the same as $SL_1(D_{\sigma(a),\sigma(b)}(\mathbb{R}))$. Therefore, we just need to understand the real quaternion algebras.

Proposition 37. *There are exactly two isomorphism classes of real quaternion algebras: the standard quaternions* $\mathbb{H}$ *and the matrix algebra* $M_2(\mathbb{R})$. *The real quaternion algebra* $D_{a,b}(\mathbb{R})$ *is isomorphic to* $\mathbb{H}$ *if* a *and* b *are both negative and otherwise it is isomorphic to* $M_2(\mathbb{R})$.

If $D_{a,b}$ is a quaternion algebra over a number field k and for some place σ of k, the real quaternion algebra $D_{\sigma(a),\sigma(b)}$ is isomorphic to $\mathbb{H}$ (resp. to $M_2(\mathbb{R})$) we say that $D_{a,b}$ is *ramified* (resp. *split*) at the place σ. So, in order for $D_{a,b}$ to give us an arithmetic subgroup of $SL_2(\mathbb{R})$, we need it to ramify at all but one place, which means we need $\sigma(a)$ and $\sigma(b)$ to both be negative at all but one place. It's easy to choose a and b satisfying this; for the typical element of $c \in k^*$, all of its images in $\mathbb{R}$ will be distinct, so for some choice of $\lambda \in \mathbb{R}$, the element $c + \lambda$ will be positive at a unique place.

The next question is when $(k, D_{a,b})$ and $(k', D_{a',b'})$ produce commensurable arithmetic groups. At least for $SL_2(\mathbb{C})$, nonisomorphic quaternion algebras always produce different commensurability classes:

Theorem 38. *([10], Theorem 8.4.1) Let* Γ_1 *and* Γ_2 *be arithmetic subgroups of* $SL_2(\mathbb{C})$. *Then* Γ_1 *is conjugate to a subgroup commensurable with* Γ_2 *if and only if* Γ_1 *and* Γ_2 *arise from the same linear algebraic group over the same number field.*

Remark 39. *I'm guessing this holds for* $SL_2(\mathbb{R})$ *as well. In fact, I think it should for every semisimple Lie group, in the sense that the arithmetic subgroup essentially uniquely determines the algebraic group over* $\mathbb{Q}$. *I think this follows from super-rigidity for groups of real rank at least two, but it really seems like it shouldn't be that hard.*

Unfortunately, the pair of numbers (a, b) does not determine the quaternion algebra uniquely up to isomorphism. A better way to classify quaternion algebras is by their ramification. We have seen that at every infinite place of k, the quaternion algebra $D_{a,b}$ is either ramified or split. The same phenomenon happens at what are called the 'finite places' of k.

Definition 40. *A finite place of a number field* k *is an embedding of* k *into a finite extension of the* p*-adic numbers for some prime integer* p.

Henceforth we will call conjugacy equivalency classes of embeddings into $\mathbb{C}$ and $\mathbb{R}$ the *infinite places* of k, and the term *place* will refer to both finite and infinite places. Definition (50) will give a uniform definition of place and we'll see that the finite places are in bijection with the prime ideals of $\mathcal{O}_k$.

If k_p is a finite extension of the p-adic numbers and $\sigma : k \to k_p$ is a finite place, then the quaternion algebra $D_{a,b}$ extends to a quaternion algebra over the field k_p. The following proposition is the analog of Proposition (37):

Proposition 41. *([10], Corollary 2.6.4) For any prime* p *and any* p*-adic field* k_p, *there are exactly two isomorphism classes of quaternion algebras over* k_p: *the matrix algebra* M_2 *and another one which we denote* $D_{u,\pi}$.

We say that $D_{a,b}$ ramifies (resp. splits) at the place σ if the quaternion algebra $D_{\sigma(a),\sigma(b)}$ is isomorphic over k_p to $D_{u,\pi}$ (resp. to M_2).

Let $\mathrm{Ram}(D)$ denote the set of places, both infinite and finite, at which D ramifies. If D has any complex places, then the ramification set obviously can't contain any of those places. Let P^* denote the set of all places of k excluding the complex places.

Theorem 42. *([10], Theorem 7.3.6) The map $D \mapsto \mathrm{Ram}(D)$ gives a bijection between isomorphism classes of quaternion algebras over k and finite subsets of P^* of even cardinality.*

Example 43. *The quaternion algebra $D_{2,3}$ from Example (18) ramifies at 2 and 3, but the correspondence is not always so obvious.*

Clearly, Theorem (42) gives a very handy description of the quaternion algebras over k. For instance, if we want to describe a quaternion algebra whose associated Lie group is $SL_2(\mathbb{R}) \times SU(2) \times \cdots \times SU(2)$, we just need to choose S to contain all but one of the real places.

One nice thing about all the work we've done is that it translates immediately to $SL_2(\mathbb{C})$. The difference is that we need to find all k-forms $\mathbf{G}$ of SL_2 such that

$$\mathrm{Res}_{k/\mathbb{Q}}\mathbf{G}(\mathbb{R}) \cong SL_2(\mathbb{C}) \times SU(2) \times \cdots \times SU(2)$$

which tells us that k must have exactly one complex place. Thus, if we replace the condition that k be totally real with the condition that k have exactly one complex place, then everything else translates immediately.

2 Volumes

We now return to our main question of computing covolumes. We have restricted ourselves to arithmetic lattices in semisimple groups, but in fact this is redundant; by an important theorem of Borel and Harish-Chandra, every arithmetic subgroup of a semisimple Lie group has finite covolume. Since arithmetic groups are easily seen to be discrete, this means every arithmetic group in a semisimple group is a lattice. We state here a more general statement of the theorem. A character is a homomorphism of algebraic groups to GL_1.

Theorem 44. *(Borel and Harish-Chandra, [2]) Let $\mathbf{G}$ be an algebraic group over $\mathbb{Q}$ such that the identity component of $\mathbf{G}$ has no nontrivial characters defined over $\mathbb{Q}$. Then any arithmetic subgroup Γ of $\mathbf{G}(\mathbb{R})$ has finite covolume.*

A semisimple group has no nontrivial characters even over $\mathbb{R}$, so this implies the finiteness for semisimple groups. Also, even though the statement of the theorem singles out the field $\mathbb{Q}$, it obviously still applies to groups over $\mathbb{Q}$ constructed by restriction of scalars from other fields k. The proof involves constructing a 'coarse fundamental domain' of finite volume for the action of Γ on $\mathbf{G}(\mathbb{R})$, which means an open set Ω in $\mathbf{G}(\mathbb{R})$ such that $\Gamma \cdot \Omega = G(\mathbb{R})$ and the set $\{\gamma \in \Gamma : \gamma\Omega \cap \Omega \neq \emptyset\}$ is finite. It's worth pointing out that the construction of a coarse fundamental domain gives more than just finiteness of the volume; it lets you use geometric group theory to do lots of nice things. For instance, it follows that all arithmetic groups are finitely presented.

We begin with an elementary computation, taken from [5], of the volume of $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$.

2.1 Covolume of $SL_n(\mathbb{Z})$

Throughout this subsection, we will use the abbreviations $G = SL_n(\mathbb{R})$ and $\Gamma = SL_n(\mathbb{Z})$ when convenient. Take $n \geq 2$. In the case of SL_n, all the analysis we need is contained in the Poisson summation formula, which we now recall. Let f be a Schwartz function on $\mathbb{R}^n$ and let

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y \rangle} dx$$

be its Fourier transform. The Poisson summation formula says

$$\sum_{x \in \mathbb{Z}^n} f(x) = \sum_{y \in \mathbb{Z}^n} \hat{f}(y).$$

Since f is a Schwartz function, so is $\hat{f}$, so both sides of this formula are finite and converge absolutely. For this argument we can choose an almost arbitrary Schwartz function as an auxilliary function - one small condition on f will arise naturally. However, convergence arguments are simpler if we assume that both f and its Fourier transform are non-negative, so we assume this.

Next we modify this formula under the action of G on $\mathbb{R}^n$. To be consistent with the rest of this paper, we will consider G acting on $\mathbb{R}^n$ on the right. For the moment, define $f_g(x) = f(xg)$ for $g \in SL_n(\mathbb{R})$. Then

$$\begin{aligned} \hat{f}_g(y) &= \int_{\mathbb{R}^n} f(xg) e^{-2\pi i \langle x, y \rangle} dx \\ &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x g^{-1}, y \rangle} \det(g^{-1}) dx \\ &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y (g^{-1})^T \rangle} dx \qquad \text{since } \det(g^{-1}) = 1 \\ &= \hat{f}(y (g^{-1})^T) \end{aligned}$$

Applying Poisson summation to f_g gives the generalized formula

$$\sum_{x \in \mathbb{Z}^n} f(xg) = \sum_{y \in \mathbb{Z}^n} \hat{f}(y (g^{-1})^T). \tag{5}$$

Equivalently we can think of G as acting on the lattice $\mathbb{Z}^n$:

$$\sum_{x \in \mathbb{Z}^n \cdot g} f(x) = \sum_{y \in \mathbb{Z}^n \cdot (g^{-1})^T} \hat{f}(y).$$

Our formula for the volume will come from integrating both sides of this equation as g varies over the group $\Gamma \backslash G$. To emphasize that we want to think of this as a family of formulas depending on g, we'll swap our script with our subscript and define

$$\vartheta_f(g) := \sum_{x \in \mathbb{Z}^n} f(xg)$$

so that equation (5) reads

$$\vartheta_f(g) = \vartheta_{\hat{f}}((g^{-1})^T). \tag{6}$$

Before we integrate over $\Gamma\backslash G$, we really ought to specify a normalization of the left Haar measure on G. As mentioned in the introduction, this is equivalent to choosing a volume form on the Lie algebra, which in this case is the space of traceless $n \times n$ matrices. The form we choose is the one so that the lattice of *integral* traceless matrices has determinant 1. We remark that the existence of this lattice is related to the fact that we can view SL_n as a group over $\mathbb{Z}$. An alternate description of this volume form is the form ω such that $\omega \wedge d(tr(A))$ is the canonical form on $M_{n\times n}$. We refer to the resulting volume form on G as dg.

Next we integrate ϑ_f over $\Gamma\backslash G$. Using the fact that the sum converges absolutely for a fixed g, we have

$$\begin{aligned}\int_{\Gamma\backslash G} \vartheta_f(g)dg &= \int_{\Gamma\backslash G} \sum_{x\in\mathbb{Z}^n} f(xg)dg \\ &= \int_{\Gamma\backslash G} f(0)dg + \int_{\Gamma\backslash G} \sum_{\substack{x\in\mathbb{Z}^n \\ x\neq 0}} f(xg)dg \qquad (7)\end{aligned}$$

The first integral on the right is just $f(0)$ times the volume of $\Gamma\backslash G$. To compute the second integral, we regroup the terms of the sum again as follows. Let $\mathbb{Z}^n_{\text{prim}}$ denote the primitive lattice points in $\mathbb{Z}^n$, i.e. those which are not a prime multiple of another lattice point. Then $\mathbb{Z}^n$ can be partitioned as the union

$$\bigcup_{l\geq 1} l \cdot \mathbb{Z}^n_{\text{prim}}.$$

Now observe that the group $SL_n(\mathbb{Z})$ acts transitively on $\mathbb{Z}^n_{\text{prim}}$. If e_n denotes the nth standard basis vector, then the stabilizer of e_n in $SL_2(\mathbb{Z})$ is the group Γ' of all matrices of the form

$$\Gamma' := \begin{bmatrix} & & & \\ & * & & * \\ & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix} \qquad (8)$$

Then $\mathbb{Z}^n_{\text{prim}} = \{e_n \cdot \gamma : \gamma \in (\Gamma'\backslash\Gamma)\}$. Therefore, we may rewrite the second integral in equation (7):

$$\int_{\Gamma\backslash G} \sum_{\substack{x\in\mathbb{Z}^n \\ x\neq 0}} f(xg)dg = \int_{\Gamma\backslash G} \sum_{l=1}^{\infty} \sum_{\gamma\in(\Gamma'\backslash\Gamma)} f(l\cdot e_n \cdot \gamma g)dg$$

We now switch the sum and the integral, which will be justified a posteriori when we show that this quantity is finite,

$$\sum_{l=1}^{\infty} \int_{\Gamma\backslash G} \sum_{\gamma\in(\Gamma'\backslash\Gamma)} f(l\cdot e_n \cdot \gamma g)dg \qquad (9)$$

and combine the second summation with the integral:

$$\sum_{l=1}^{\infty} \int_{\Gamma'\backslash G} f(l\cdot e_n \cdot g)dg. \qquad (10)$$

Let's now study the domain $\Gamma'\backslash G$. Let G' be the group of matrices of the form (8) but with real coefficients, i.e. the stabilizer of e_n in G. Since G acts transitively on $\mathbb{R}^n \setminus \{0\}$, it follows that the homogeneous space $G'\backslash G$ is exactly $\mathbb{R}^n \setminus \{0\}$. The fibers of the map $\Gamma'\backslash G \to G'\backslash G$ are naturally identified with $\Gamma'\backslash G'$, so $\Gamma'\backslash G$ is the total space of a fiber bundle

$$\begin{array}{ccc} \Gamma'\backslash G' & \longrightarrow & \Gamma'\backslash G \\ & & \downarrow \pi \\ & & \mathbb{R}^n \setminus \{0\} \end{array}$$

Observe that the integrand in equation (10) is constant on the fibers of this fibration. This suggests that we compute the integral by pushing it forward to $\mathbb{R}^n \setminus \{0\}$.

Proposition 45. *The pushforward of the measure dg under the map $\pi : \Gamma'\backslash G \to \mathbb{R}^n \setminus \{0\}$ is G-invariant.*

Proof. Let U be a measurable set in $\mathbb{R}^n \setminus \{0\}$. We need to show that $\pi_* dg(U) = \pi_* dg(Uh)$ for any $h \in G$. Thinking of U as a set of right cosets of G' in G, we have

$$\pi_* dg(U) = \int_{\pi^{-1}(U)} dg = \int_{G'g \in U} \int_{\Gamma\backslash G' \ni g'} dg(g'g). \tag{11}$$

Since dg is left-invariant, it descends to a measure on $\Gamma\backslash G$, so the expression $dg(g'g)$ makes sense even though $g'g$ is only well-defined mod Γ. On the other hand

$$\begin{aligned} \pi_* dg(Uh) &= \int_{G'g \in Uh} \int_{\Gamma\backslash G' \ni g'} dg(g'g) \\ &= \int_{G'gh^{-1} \in U} \int_{\Gamma\backslash G' \ni g'} dg(g'g) \\ &= \int_{G'\widetilde{g} \in U} \int_{\Gamma\backslash G' \ni g'} dg(g'\widetilde{g}h) \end{aligned} \tag{12}$$

where we've substituted $\widetilde{g} = gh^{-1}$. Now the proof boils down to the fact that the 'modular character' of G is trivial. Recall that the modular character χ_G of G is a homomorphism $G \to \mathbb{R}^+$ defined implicitly by

$$R_h^* \mu_G = \chi_G(h) \mu_G$$

where μ_G is a left Haar measure on G and R_h means multiplication on the right by h. Since $G = SL_n$ is simple every character is trivial, in particular the modular one. Therefore $dg(g'\widetilde{g}h) = dg(g'\widetilde{g})$, and comparing Equations (11) and (12) shows that the pushforward of dg is G-invariant. □

The restriction of the Lebesgue measure dx from $\mathbb{R}^n$ to $\mathbb{R}^n \setminus \{0\}$ is also G-invariant (remember, $G = SL_n$). But invariant measures on homogeneous spaces are unique up to a global rescaling, so we conclude that $\pi_* dg = \lambda dx$.

Now let dg' be a left Haar measure on G' such that on the Lie algebra of G we have

$$dg = dh \wedge \pi^* dx \tag{13}$$

(please excuse my failure to distinguish the measure and the corresponding form). We'll see in a minute (Proposition (46)) that G' is unimodular as well; combined with the observation that $\pi^* dx$ is also invariant under right multiplication by G', this shows that the identity (13) holds at every point of G'. It follows that

$$\lambda := \pi_* dg/dx = \mathrm{vol}_{dg'}(\Gamma' \backslash G').$$

There's a nice trick to determine the correct normalization of dg'. Recall that we defined dg to be the Haar measure that gave the $\mathbb{Z}$-lattice in the Lie algebra volume one. Lebesgue measure on $\mathbb{R}^n$ is also normalized to give the lattice $\mathbb{Z}^n$ volume one. It's an easy check that the identification

$$\mathrm{Lie}(G)/\mathrm{Lie}(G') \cong T_{e_n}(\mathbb{R}^n \setminus \{0\})$$

maps the $\mathbb{Z}$-lattice in $\mathrm{Lie}(G)$ to the $\mathbb{Z}$-lattice in $T_{e_n}(\mathbb{R}^n \setminus \{0\})$. Hence, we see that we should normalize dg' so that the lattice

$$\mathrm{Lie}(G')(\mathbb{Z}) = \mathrm{Lie}(G)(\mathbb{Z}) \cap \mathrm{Lie}(G')$$

has volume one.

Returning to equation (10), integrating first with respect to dg' gives

$$\begin{aligned}
\int_{\Gamma' \backslash G} f(l \cdot e_n \cdot g) dg &= \int_{\mathbb{R}^n \backslash \{0\}} \int_{\Gamma' \backslash G'} f(l \cdot e_n \cdot g' g) dg' dx \\
&= \mathrm{vol}(\Gamma' \backslash G') \int_{\mathbb{R}^n \backslash \{0\}} f(lx) dx \\
&= \mathrm{vol}(\Gamma' \backslash G') \int_{\mathbb{R}^n} f(lx) dx \\
&= \frac{1}{l^n} \mathrm{vol}(\Gamma' \backslash G') \int_{\mathbb{R}^n} f(x) dx \\
&= \frac{1}{l^n} \mathrm{vol}(\Gamma' \backslash G') \hat{f}(0) \qquad (14)
\end{aligned}$$

We can compute the volume of $\Gamma' \backslash G'$ with the same ideas. Recall that G' is the set of matrices of the form

$$\begin{bmatrix} & h & & \vec{x} \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

where $h \in SL_{n-1}(\mathbb{R})$ and $\vec{x} \in \mathbb{R}^{n-1}$. If we think of this group as now acting on column vectors on the left instead of row vectors on the right, so that a vector $\vec{v}$ gets sent to $h\vec{v} + \vec{x}$, we see that it is exactly the special affine group of $\mathbb{R}^{n-1}$. In other words, it is a semi-direct product

$$0 \to SL_{n-1}(\mathbb{R}) \to G' \to \mathbb{R}^{n-1} \to 0$$

Moreover Γ' is also a semidirect product:

$$0 \to SL_{n-1}(\mathbb{Z}) \to \Gamma' \to \mathbb{Z}^{n-1} \to 0$$

Therefore $\Gamma'\backslash G'$ is the total space of a fiber bundle

$$\begin{array}{ccc} SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R}) & \longrightarrow & \Gamma'\backslash G' \\ & & \downarrow \\ & & \mathbb{Z}^n\backslash\mathbb{R}^n \end{array}$$

We can apply the argument in Proposition (45) to compute the volume as long as G' is unimodular.

Proposition 46. *The special affine group G' is unimodular.*

Proof. The product of the Haar measure on SL_{n-1} and the Haar measure on $\mathbb{R}^{n-1}$ is easily seen to be bi-invariant. $\square$

The conclusion, as in Proposition (45), is that if dx is an invariant volume form on $\mathbb{R}^n$ and dg is an invariant volume form on $SL_{n-1}(\mathbb{R})$ such that $dg' = dg \wedge dx$ on the Lie algebra of G', then $dg' = dg \wedge dx$ everywhere. Since the map of Lie algebras still preserves the $\mathbb{Z}$-sublattice, we can take dx to be the Lebesgue measure on $\mathbb{R}^n$ and dg to be normalized as it was for SL_n. We conclude

$$\mathrm{vol}(\Gamma'\backslash G') = \mathrm{vol}(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R})).$$

Therefore, the quantity (10) becomes

$$\sum_{l=1}^{\infty} \frac{1}{l^n} \mathrm{vol}(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R}))\hat{f}(0).$$

In terms of the Riemann zeta function, this is

$$\zeta(n)\mathrm{vol}(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R}))\hat{f}(0).$$

Making the inductive hypothesis that $\mathrm{vol}(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R}))$ is finite, and using the hypothesis that f is non-negative, the dominated convergence theorem shows that our interchange of the integral and the sum in Equation (9) is justified.

Now going back to equation (7) we have

$$\int_{\Gamma\backslash G} \vartheta_f(g)dg = \mathrm{vol}(\Gamma\backslash G)f(0) + \zeta(n)\mathrm{vol}(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R}))\hat{f}(0) \tag{15}$$

Let's now use the Poisson summation formula, in the form of equation (6), which we reproduce:

$$\vartheta_f(g) = \vartheta_{\hat{f}}((g^{-1})^T)$$

The map from $SL_n(\mathbb{R})$ to itself taking g to $(g^{-1})^T$ is a measure-preserving automorphism of $SL_n(\mathbb{R})$. Moreover, it restricts to an automorphism of $SL_2(\mathbb{Z})$. Together these properties imply that it gives a measure preserving map from $\Gamma\backslash G$ to itself and therefore

$$\int_{\Gamma\backslash G} \vartheta_{\hat{f}}((g^{-1})^T) = \int_{\Gamma\backslash G} \vartheta_{\hat{f}}(g)$$

Applying Equation (15) to both sides, we get

$$\begin{aligned} & \mathrm{vol}(\Gamma\backslash G)f(0) + \zeta(n)\mathrm{vol}(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R}))\hat{f}(0) \\ = \quad & \mathrm{vol}(\Gamma\backslash G)\hat{f}(0) + \zeta(n)\mathrm{vol}(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R}))f(0) \end{aligned}$$

Finally, we impose retrospective the condition on the auxiliary function f that $f(0) = 0$ but $\hat{f}(0) \neq 0$. Then we conclude.

$$\mathrm{vol}(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})) = \zeta(n)\mathrm{vol}(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R})) \tag{16}$$

Since $SL_1(\mathbb{R})$ is a point, this shows that

$$\mathrm{vol}(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})) = \zeta(2)\zeta(3)\cdots\zeta(n). \tag{17}$$

Remark 47. *Analytically, the essence of this argument is the Poisson formula. But there's also an essential algebraic trick, namely replacing the integral over $\Gamma\backslash G$ with an integral over $\Gamma'\backslash G$. Perhaps this trick is less mysterious if we give an interpretation of these spaces. Since any two lattices of determinant one in $\mathbb{R}^n$ are related by a transformation in G and Γ is the stabilizer of the lattice $\mathbb{Z}^n$, we can interpret $\Gamma\backslash G$ as the space of all unimodular lattices in $\mathbb{R}^n$. Similarly, $\Gamma'\backslash G$ is the space of all pairs (Λ, v) where Λ is a unimodular lattice in $\mathbb{R}^n$ and v is a primitive vector in Λ. The only way to understand a lattice is to look at the vectors in it, so it makes sense that it might be easier to work with the space $\Gamma'\backslash G$.*

2.2 The other groups

What enabled us to use the Poisson summation to compute the volume of $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ was the fact that $SL_n(\mathbb{R})$ acts transitively on $\mathbb{R}^n \setminus \{0\}$. Unfortunately a general Lie group does not admit a transitive action of $\mathbb{R}^n \setminus \{0\}$; the complete list of simple Lie groups which act transitively on $\mathbb{R}^n \setminus \{0\}$ is

$$SL_n(\mathbb{R}), SL_n(\mathbb{C}), SL_n(\mathbb{H}), Sp_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{C}), Spin_{9,1}(\mathbb{R})$$

([8], Theorem 6.17(b)). We can indeed use Poisson summation to calculate the covolume of $Sp_{2n}(\mathbb{Z})$ and the other examples would be interesting to think about too. We outline the differences between the calculation for Sp_{2n} and for SL_n; for details see [5].

Consider the right action of $Sp_{2n}(\mathbb{R})$ on $\mathbb{R}^{2n}$ and let G' be the stabilizer of the nth basis vector e_n; it consists of those matrices in Sp_{2n} that are of the form

$$\begin{bmatrix} & & & b_{1n} \\ & h & & \vdots \\ & & & d_{(n-1)n} \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

The argument proceeds as before, and we arrive at the analog of Equation (16):

$$\mathrm{vol}(Sp_{2n}(\mathbb{Z})\backslash Sp_{2n}(\mathbb{R})) = \zeta(n)\mathrm{vol}(G'(\mathbb{Z})\backslash G') \tag{18}$$

To find the volume of $G'(\mathbb{Z})\backslash G'$ we look at the left action of G' on $\mathbb{R}^{2n-1}$ by the formula

$$\begin{bmatrix} & h & & \vec{x} \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{v} \\ 1 \end{bmatrix} = \begin{bmatrix} h\vec{v} + \vec{x} \\ 1 \end{bmatrix}$$

The stabilizer of the origin under this action consists of matrices of the form

$$\begin{bmatrix} & & & 0 \\ & h & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

It is easiest to determine the conditions on h at the level of Lie algebras. The Lie algebra of Sp_{2n} consists of matrices of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{with } A^T = -D,\ B^T = B, \text{ and } C^T = C.$$

Hence, the Lie algebra of h consists of matrices of the form

$$\left[\begin{array}{ccc|ccc} & & 0 & & & 0 \\ \{a_{ij}\} & & \vdots & \{b_{ij}\} & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline & & 0 & & & 0 \\ \{c_{ij}\} & & \vdots & \{d_{ij}\} & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right]$$

Therefore, the stabilizer of the origin is isomorphic to $Sp_{2n-1}(\mathbb{R})$. It follows as for SL_n that $\text{vol}(G'(\mathbb{Z})\backslash G') = \text{vol}(Sp_{2(n-1)}(\mathbb{Z})\backslash Sp_{2(n-1)}(\mathbb{R}))$. So Equation (18) lets us compute inductively:

$$\text{vol}(Sp_{2n}(\mathbb{Z})\backslash Sp_{2n}(\mathbb{R})) = \zeta(2)\zeta(4)\cdots\zeta(2n). \tag{19}$$

What's the natural way to generalize this formula beyond SL_n and Sp_{2n}? First, let's look at the sequence of numbers at which we evaluate the zeta function: $(2, 3, \ldots, n)$ for $SL_n(\mathbb{Z})$ and $(2, 4, \ldots, 2n)$ for $Sp_{2n}(\mathbb{Z})$. In both cases, these numbers are what are called the characteristic exponents of the Lie group. These are defined in terms of the representation of the Weyl group on the root system. Namely, the ring of invariant polynomials of the Weyl group action is always freely generated and the characteristic exponents are defined to be the degrees of the generators. Characteristic exponents also admit several equivalent descriptions in terms of cohomology; for instance, if a_i are the characteristic exponents of G, then the numbers $2a_i + 1$ are the dimensions of the generators of the cohomology ring of the underlying manifold for the compact (or complex) form of G.

Note that these are invariants which depend only on the Cartan type of the Lie group, or equivalently only on its complexification. So if we want to generalize the formulas for SL_n and Sp_{2n} to other Cartan types, we need to somehow choose the 'right' real form as well as the 'right' arithmetic subgroup within that Cartan type. The right real form turns out to be the split form.

Definition 48. *A semisimple group* $\mathbf{G}$ *defined over a field* k *is* split *if it contains a maximal torus which is isomorphic to a direct product of copies of* GL_1*. (In this case, we also say the torus is split).*

Recall that there is a unique split real form of any complex Lie group. For the classical groups, these are listed in Table 1.

Table 1: Split forms

Cartan Type	Complex Lie Group	Split Real Lie Group
A_n	$SL_{n+1}(\mathbb{C})$	$SL_{n+1}(\mathbb{R})$
B_n	$SO_{2n+1}(\mathbb{C})$	$SO(n, n+1)$
C_n	$Sp_{2n}(\mathbb{C})$	$Sp_{2n}(\mathbb{R})$
D_n	$SO_{2n}(\mathbb{C})$	$SO(n, n)$

In fact there is also a unique split form for each Cartan type over any number field. The right generalization of our volume formulas is to groups that are split over $\mathbb{Q}$. For the right arithmetic subgroup, we use the construction of Chevalley. For each Cartan type, this is a particular lattice $\mathfrak{g}_\mathbb{Z}$ in the Lie algebra $\mathfrak{g}$ which is closed under the Lie bracket. It determines a particularly nice arithmetic subgroup of the split form of the group. We remark that Chevalley's construction gives the standard arithmetic subgroups of for the classical split groups.

In this setting, Langlands proved the following theorem.

Theorem 49. *(Langlands, [9]) Let* $\mathbf{G}$ *be a Chevalley group over* $\mathbb{Q}$*. Normalize the top form corresponding to the Haar measure* dg *on* $G(\mathbb{R})$ *so that the lattice* $\mathfrak{g}_\mathbb{Z}$ *in the Lie algebra* $\mathfrak{g}$ *has volume 1. Let* a_i *be the characteristic exponents of* $\mathbf{G}$ *and let* c *be the order of the fundamental group of* $G_\mathbb{C}$*. Then*

$$\mathrm{vol}_{dg}\mathbf{G}(\mathbb{Z})\backslash\mathbf{G}(\mathbb{R}) = c\prod_{i=1}^{p}\zeta(a_i)$$

A good exposition of this proof can be found in [17]. Langlands also remarks that if we replace $\mathbb{Q}$ by a general number field k and replace the zeta function by the Dedekind zeta function of k defined as a sum over ideals of $\mathcal{O}_k$:

$$\zeta_k(s) = \sum_{I\subset\mathcal{O}_k}\frac{1}{N_{k/\mathbb{Q}}(I)^s}$$

then the corresponding statement still holds.

It turns out that though a whole lot of work it is possible to relate the covolume of every lattice in every semisimple Lie group to one of these. In order to state these deep results, we need to first introduce adeles and Tamagawa numbers. To motivate these, we begin by counting the points of SL_n over finite fields.

2.3 The p-adic volumes of SL_n

Consider the action of $SL_n(\mathbb{F}_p)$ on $\mathbb{F}_p^n$, where $\mathbb{F}_p$ is the field with p elements. To be consistent with the rest of this paper, we think of it acting on the right. As with the real numbers, $SL_n(\mathbb{F}_p)$ acts transitively on the nonzero vectors of $\mathbb{F}_p^n$ and the stabilizer of e_n is the subgroup of all matrices of the form

$$\begin{bmatrix} & h & & \vec{x} \\ & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

with $h \in SL_{n-1}(\mathbb{F}_p)$. Since the numbers in the rightmost column can be arbitrary, the size of this subgroup is $p^{n-1}|SL_{n-1}(\mathbb{F}_p)|$. Multiplying this by the number of nonzero vectors in $\mathbb{F}_p^n$ gives

$$|SL_n(\mathbb{F}_p)| = (p^n - 1)p^{n-1}|SL_{n-1}(\mathbb{F}_p)|.$$

Since $|SL_1(\mathbb{F}_p)| = 1$, we have

$$|SL_n(\mathbb{F}_p)| = \prod_{k=2}^{n} (p^k - 1)p^{k-1}$$

Now here is the magic: if we normalize each of these counts by dividing by $p^{\dim(SL_n)}$ and multiply them together over all primes p, we get

$$\begin{aligned} \prod_p \frac{1}{p^{n^2-1}} \prod_{k=2}^{n} (p^k - 1)p^{k-1} &= \prod_p \prod_{k=2}^{n} (1 - \frac{1}{p^k}) \\ &= \prod_{k=2}^{n} \zeta(k)^{-1}. \end{aligned} \tag{20}$$

Combining this with Equation (17) gives

$$SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R}) \prod_p \frac{|SL_n(\mathbb{F}_p)|}{p^{\dim(SL_n)}} = 1. \tag{21}$$

We seem to be able to compute the covolume of $SL_n(\mathbb{Z})$ by counting points over finite fields instead of by integrating. The rest of this section and the next will be devoted to explaining the formula (21).

First we interpret it in a way which is more coherent, if equally mysterious, by thinking of the normalized count of points mod p as the volume of the integral p-adic points of SL_n. We recall the construction of the p-adics and their extensions.

Let k be a number field and $\mathfrak{p}$ a prime ideal of its ring of integers $\mathcal{O}_k$. Since every nonzero element of $\mathcal{O}_k$ generates an ideal which factors uniquely into a product of prime ideals, there is a well defined 'valuation' $v_\mathfrak{p} : \mathcal{O}_k \setminus \{0\} \to \mathbb{Z}^{\geq 0}$ which picks out the power of $\mathfrak{p}$ in the prime factorization. The valuation $v_\mathfrak{p}$ extends naturally to a map from $k \setminus \{0\}$ to $\mathbb{Z}$, which we also call $v_\mathfrak{p}$. We then define the $\mathfrak{p}$-adic absolute value

$$|a|_\mathfrak{p} := N(\mathfrak{p})^{-v_\mathfrak{p}(a)/d} \qquad a \in k$$

where $N(\mathfrak{p}) = |\mathcal{O}_k/\mathfrak{p}|$ is the norm of the ideal $\mathfrak{p}$ and d is the degree of k over $\mathbb{Q}$. The $\mathfrak{p}$-adic numbers $k_\mathfrak{p}$ are defined as a metric space to be the completion of k with respect to the metric induced by this absolute value; one checks that $k_\mathfrak{p}$ naturally inherits the structure of a field. The valuation and the absolute value $|\cdot|_\mathfrak{p}$ extend to $k_\mathfrak{p}$. The field $k_\mathfrak{p}$ is a finite extension of a corresponding p-adic field $\mathbb{Q}_p$, and the inclusions $k \hookrightarrow k_\mathfrak{p}$ are exactly the finite places defined in (40).

One can define an absolute value on k abstractly a metric on k which interacts well with the field structure so that the completion with respect to that metric is still a field. Then one can show that every absolute value on k, up to normalization, comes either from a $\mathfrak{p}$-adic valuation on k or from an embedding of k into $\mathbb{R}$ or $\mathbb{C}$. Therefore we can make a uniform definition which includes both the infinite and finite places:

Definition 50. *A* place *of k as an embedding of k into one of its metric completions.*

We define the ring $\mathcal{O}_\mathfrak{p}$ of $\mathfrak{p}$-adic integers to be those $\mathfrak{p}$-adic numbers with nonnegative valuation, i.e. with absolute value less than or equal to one. The ring $\mathcal{O}_\mathfrak{p}$ is compact with respect to the topology induced by the absolute value, and therefore $k_\mathfrak{p}$ is locally compact. It follows from this that linear algebraic groups defined over the field $k_\mathfrak{p}$ are also locally compact and therefore have left and right Haar measures. On $k_\mathfrak{p}$ itself, we normalize the measure so that the $\mathfrak{p}$-adic integers have volume 1.

Now we return to considering the p-adic group $SL_n(\mathbb{Q}_p)$. We define $SL_n(\mathbb{Z}_p)$ to be the stabilizer of the lattice $\mathbb{Z}_p^n$ inside $SL_n(\mathbb{Q}_p)$. We could also think of $SL_n(\mathbb{Z}_p)$ as the $\mathbb{Z}_p$-points of the group scheme SL_n defined over $\mathbb{Z}$. By the previous paragraph, $SL_n(\mathbb{Q}_p)$ has a natural one-parameter family of left Haar measures, but we don't yet have a very good description of them. In order to relate the volume of $SL_n(\mathbb{Z}_p)$ to the volume of $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ like we do in Equation (21), we'll need some way to relate the Haar measure on $SL_n(\mathbb{Q}_p)$ to the Haar measure on $SL_n(\mathbb{R})$. The way to do this is through top forms on the Lie algebra.

We know that real-valued top-forms on the Lie algebra $\mathfrak{sl}_n(\mathbb{R})$ correspond exactly to Haar measures on $SL_n(\mathbb{R})$. But there's subset of these top-forms that are really more natural: the top forms defined over $\mathbb{Q}$. Note that a top form on a vector space V is a point of the top exterior power of the dual of V, so if we think of V as an algebraic object defined over a field k then we can think of top-forms on V as points over a k-algebra, and we can apply our whole framework of algebraic objects and their sets of points. In particular, if a top form is defined over $\mathbb{Q}$, then it also determines a top form on $\mathfrak{sl}_n(\mathbb{Q}_p)$ for any p.

Recall that there was a fairly natural choice of top-form ω on $\mathfrak{sl}_n(\mathbb{R})$: that which gives the lattice $\mathfrak{sl}_n(\mathbb{Z})$ covolume one. This form, which we'll call ω, lives over $\mathbb{Q}$ so it determines a top form on $\mathfrak{sl}_n(\mathbb{Q}_p)$. This p-adic top form also has a nice description in terms of the integral lattice. Define $\mathfrak{sl}_n(\mathbb{Z}_p)$ in the natural way, either as the $\mathbb{Z}_p$-points of the scheme $\mathfrak{sl}_n$ over $\mathbb{Z}$ or as the stabilizer of $\mathbb{Z}_p^n$. Then I claim that the volume of $\mathfrak{sl}_n(\mathbb{Z}_p)$ with respect to ω is one. Let $A_1, \ldots, A_r$ be a basis for the lattice $\mathfrak{sl}_n(\mathbb{Z})$, where $r = n^2 - 1$ is the dimension of $\mathfrak{sl}_n$. Let $\omega_1, \ldots, \omega_r$ be a dual basis, so that

$$\omega = \omega_1 \wedge \cdots \wedge \omega_r.$$

The basis $A_1, \ldots, A_r$ gives an isomorphism of $\mathfrak{sl}_n(\mathbb{Q}_p)$ with $\mathbb{Q}_p^r$ sending $\mathfrak{sl}_n(\mathbb{Z})$ to $\mathbb{Z}_p^r$ and ω to the standard volume form $dx_1 \wedge \cdots \wedge dx_r$ on $\mathbb{Q}_p^r$. Since the integral of dx over $\mathbb{Z}_p$ is defined to be one, this proves the claim.

Now we want to see how the form ω on $\mathfrak{sl}_n(\mathbb{Q}_p)$ determines a particular Haar measure on $SL_n(\mathbb{Q}_p)$. The general construction is briefly presented below, but for now the following intuition will be sufficient: the exponential map should be infinitessimally volume-preserving, so we should have

$$\lim_{\epsilon\to 0} \frac{\mathrm{vol}_{\mathrm{Haar}}(\exp(B_\epsilon(0)))}{\mathrm{vol}_\omega(B_\epsilon(0))} = 1 \tag{22}$$

where $B_\epsilon(0)$ is some suitably defined notion of a ball of radius ϵ in $\mathfrak{sl}_n(\mathbb{Q}_p)$. The reason this is enough to define the Haar measure is that it turns out that for small enough values of ϵ, this ratio stabilizes. Therefore we know what the Haar measure of a small enough neighborhood of E_n should be. But any other ball in the Lie group of any radius is commensurable with a translate of the first, and since Haar measure is (left) translation invariant, this determines the measure of any ball.

In fact, we'll see for almost any prime p the ratio is independent of ϵ as long as ϵ is less than one. For any positive integer s, let $SL_n^(s)(\mathbb{Z}_p)$ be kernel of the reduction map from $SL_n(\mathbb{Z}_p)$ to $SL_n(\mathbb{Z}/p^s)$ and let $\mathfrak{sl}_n^{(s)}(\mathbb{Z}_p)$ be the kernel of the reduction map to from $\mathfrak{sl}_n(\mathbb{Z}_p)$ to $\mathfrak{sl}_n(\mathbb{Z}/p^s)$.

If we use the L^∞ metric on $n\times n$ matrices, i.e. that which takes the supremum of the absolute value of their coordinates, then $\mathfrak{sl}_n^{(s)}(\mathbb{Z}_p) = B_{p^{-s}}(0)$ and $SL_n^{(s)}(\mathbb{Z}_p) = B_{p^{-s}}(E_n)$. Since $\mathfrak{sl}_n^{(-s)}(\mathbb{Z}_p)$ has index p^{rs} in $\mathfrak{sl}_n(\mathbb{Z}_p)$ and the latter has volume one, we have

$$\mathrm{vol}_\omega(B_{p^{-s}}(0)) = \mathrm{vol}_\omega(\mathfrak{sl}_n^{(s)}(\mathbb{Z}_p)) = \frac{1}{p^{rs}}. \tag{23}$$

We can do a similar analysis of the numerator. Where the expansion of the exponential map

$$\exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!}$$

and the logarithm

$$\log(E_n - g) = -\sum_{i=1}^{\infty} \frac{g^i}{i}$$

are both p-adically convergent, they are inverse to one another. By counting the number of powers of p in $i!$, we can see that the logarithm series is convergent for $g \in B_\epsilon(E_n)$, $\epsilon < 1$, and the exponential series is convergent for $A \in B_\epsilon(0)$, $\epsilon < 1$, except if $p = 2$ in which case we need to take $\epsilon < 1/2$. It follows that the exponential map defines an isomorphism

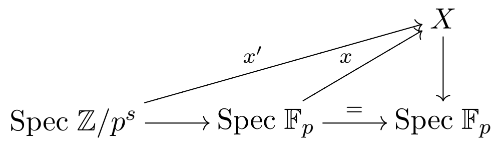
$$\exp : \mathfrak{sl}_n^{(s)}(\mathbb{Z}_p) \xrightarrow{\cong} SL_n^{(s)}(\mathbb{Z}_p)$$

for every $s \geq 1$ (with the silly exception for $p = 2$).

Having identified the open set $\exp(B_\epsilon(0))$ with a congruence subgroup, we can deduce its volume using Hensel's lemma.

Proposition 51. *(Hensel's lemma) Let X be a a scheme defined of dimension r over the ring $\mathbb{Z}_p$, and let $X_{F_\mathfrak{p}}$ be the extension of scalars to the finite field $\mathbb{F}_\mathfrak{p}$. Let x be a smooth $\mathbb{F}_p$-point of X. If $X = \mathrm{Spec}\ k[x_1, \ldots, x_{r+1}]/(f)$, this just means that the derivative of f at*

x is not divisible by p. Then for any positive integer s, there are exactly p^{rs} $\mathbb{Z}/p^s$-points x' of $X_{F_\mathfrak{p}}$ lying over x in the sense that the diagram commutes:



Since SL_n is defined by the equation $\det = 1$ and the coefficients of the determinant are all plus or minus one, SL_n is smooth at every prime p, so Hensel's lemma is applicable.

Proof. We'll do the case $X = \text{Spec } k[x_1, \ldots, x_{r+1}]/(f)$. Then we can think of a point $x \in X$ as an $r+1$-dimensional vector. Suppose we have already lifted x to a point over $\mathbb{Z}/p^s$. The lifts of a point x from $\mathbb{Z}/p^s$ to $\mathbb{Z}/p^{s+1}$ are the solutions $y \in (\mathbb{Z}/p)^{r+1}$ to

$$f(x + p^r y) \cong 0 \mod p^{r+1}.$$

You can check by hand that

$$f(x + p^r y) \cong f(x) + f'(x)p^r y \mod p^{r+1}.$$

Since x is a $\mathbb{Z}/p^s$ point of X, $f(x)$ is divisibly by p^s but not necessarily by p^{s+1}. But if $f'(x)$ is not divisibly by p, then it is a surjective map from $(\mathbb{Z}/p)^{r+1}$ to $\mathbb{Z}/p$, so no matter what $f(x)$ is there are exactly p^r solutions for y. Iterating this lifting process s times gives the theorem. □

If we take the limit as s goes to infinity, Hensel's lemma shows that every $\mathbb{F}_p$ point of X is the reduction of many $\mathbb{Z}_p$ points of X. In fact, it says how many. Namely, the number of lifts of x from $\mathbb{Z}/p^s$ to $\mathbb{Z}/p^{s+1}$ is exactly the index

$$[SL_n^{(s)}(\mathbb{Z}_p) : SL_n^{(s+1)}(\mathbb{Z}_p)].$$

Therefore, as epsilon decreases from p^{-s} to $p^{-(s+1)}$, the numerator of Equation (22) decreases by a factor of p^r. This is the same factor by which the denominator decreases, so in this sense the ratio is stable for $\epsilon < 1$. This determines the Haar measure on $SL_n(\mathbb{Q}_p)$ and shows that the volume of $SL_n(\mathbb{Z}_p)$ is given by

$$\text{vol}_{\text{Haar}}(SL_n(\mathbb{Z}_p)) = \frac{|SL_n(\mathbb{Z}/p)|}{\text{vol}_{Haar}(SL_n^{(1)}(\mathbb{Z}_p))} = \frac{|SL_n(\mathbb{Z}/p)|}{p^r}.$$

A priori if $p = 2$ we only know that this works for $SL_n^{(2)}(\mathbb{Z}_2)$, but you can easily check that even for $p = 2$ it actually works for $SL_n^{(1)}$.

This shows that our normalized count of points mod p really is a p-adic volume, so we can restate Equation (21) as

$$\text{vol}_\omega(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})) \prod_p \text{vol}_\omega(SL_n(\mathbb{Z}_p)) = 1 \tag{24}$$

2.4 Adeles and Tamagawa numbers

In this section we will see how to generalize formula (24) to groups other than SL_2. Because the number theory involved gets rapidly more complicated, we'll state most theorems without proofs. We'll see that there is an analog of this formula for any semisimple linear algebraic group $\mathbf{G}$, but not for any arithmetic subgroup. This is where it becomes important that one work with group schemes over $\mathcal{O}_k$; in our case for simplicity we restrict further to arithmetic subgroups that are of the form $\mathbf{G}(\mathcal{O}_k)$ for some representation of $\mathbf{G}$ on k^n.

Recall that if $\mathbf{G} \to GL_n$ is a representation of a group $\mathbf{G}$ defined over k, we define

$$\mathbf{G}(\mathcal{O}_k) = \mathbf{G}(k) \cap GL_n(\mathcal{O}_k).$$

As we did for SL_n, for any prime ideal $\mathfrak{p} \subset k$, define $\mathbf{G}(\mathcal{O}_\mathfrak{p})$ to be the stabilizer of the lattice $\mathcal{O}_\mathfrak{p}^n \subset k_\mathfrak{p}^n$.

The representation of $\mathbf{G}$ also lets us define the variety $\mathbf{G}_{\mathbb{F}_\mathfrak{p}}$ over the finite field $\mathbb{F}_\mathfrak{p}$. Since we're avoiding group schemes, we'll do this by hand. The image of $\mathbf{G}$ in GL_n is the vanishing locus of finitely many polynomials $f_1, \dots, f_d$ over k. Since the image contains the point E_n, we can assume that every polynomial f_i has integral coefficients by rescaling about E_n. Then we define the variety $\mathbf{G}_{\mathbb{F}_\mathfrak{p}}$ to be

$$\operatorname{Spec} \mathbb{F}_p[x_{11}, \dots, x_{nn}]/(f_1, \dots, f_d).$$

We only defined linear algebraic groups over fields of characteristic zero, so we won't concern ourselves with the question of whether $\mathbf{G}_{\mathbb{F}_\mathfrak{p}}$ is a linear algebraic group. But since $\mathbf{G}$ is a smooth variety over k, an argument using the resultant shows that it $\mathbf{G}_{\mathbb{F}_\mathfrak{p}}$ is a smooth variety for all but finitely many primes. For primes at which it's smooth, Hensel's lemma applies and we can use the same argument as for SL_n to show that

$$\mathrm{vol}_{\mathrm{Haar}} \mathbf{G}(\mathcal{O}_\mathfrak{p}) = \frac{|G(\mathbb{F}_\mathfrak{p})|}{N(\mathfrak{p})^{\dim(\mathbf{G})}} \, \mathrm{vol}_\omega \mathfrak{g}(\mathcal{O}_\mathfrak{p}). \tag{25}$$

where $\mathfrak{g}(\mathcal{O}_\mathfrak{p})$ stabilizer of $\mathcal{O}_\mathfrak{p}^n$ in $\mathfrak{g}(k_\mathfrak{p})$. At the exceptional primes, however, this doesn't work

Example 52. *The algebraic group $SL_1(D_{2,3})$ over $\mathbb{Q}$ is smooth over every prime except for 2 and 3 with respect to the matrix representation (3). It's defined by the equation*

$$w^2 - 2x^2 - 3y^2 + 6z^2 = 1$$

along with some unimportant linear equations defining the algebra $D_{2,3}$ as a subalgebra of the matrix algebra M_4. The point $[1, 1, 0, 0]$ is a point on this variety modulo 2, but there is no point modulo 4 lying over it.

However, it turns out that even at these bad primes, the limit (22) stabilizes over even smaller balls and so we can still construct a Haar measure on $\mathbf{G}(k_\mathfrak{p})$ from a form on its Lie algebra $\mathfrak{g}(k_\mathfrak{p})$. We just need to take a small enough ϵ. As we did for $SL_n^{(s)}(\mathbb{Z}_p)$, define $\mathbf{G}^{(s)}(\mathcal{O}_\mathfrak{p})$ to be the kernel of the reduction map from $\mathbf{G}(\mathcal{O}_\mathfrak{p})$ to $\mathbf{G}(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^s)$.

Theorem 53. *([17], Theorem 9.2) At any prime $\mathfrak{p}$, there exists an s_0 such that for any $s \geq s_0$,*

$$\mathrm{vol}_{Haar}\mathbf{G}^{(s)}(\mathcal{O}_{\mathfrak{p}}) = N(\mathfrak{p})^{-s\dim(\mathbf{G})}\mathrm{vol}_{\omega}\mathfrak{g}(\mathcal{O}_{\mathfrak{p}})$$

and therefore

$$\mathrm{vol}_{Haar}\mathbf{G}(\mathcal{O}_p) = \frac{|\mathbf{G}(\mathcal{O}_p)/\mathbf{G}^{(s)}(\mathcal{O}_p)|}{N(\mathfrak{p})^{s\dim(\mathbf{G})}}\,\mathrm{vol}_{\omega}\mathfrak{g}(\mathcal{O}_p). \tag{26}$$

Note that although $\mathbf{G}(\mathcal{O}_p)/\mathbf{G}^{(s)}(\mathcal{O}_p)$ maps to $\mathbf{G}_{\mathbb{F}_{\mathfrak{p}}}(\mathcal{O}/p^s)$, Example (52) shows that this map is not necessarily surjective, so unlike in the smooth case, we can't replace the numerator with $|\mathbf{G}_{\mathbb{F}_{\mathfrak{p}}}(\mathcal{O}/p^s)|$. This is unfortunate because it's much easier to calculate the size of $\mathbf{G}_{\mathbb{F}_{\mathfrak{p}}}(\mathcal{O}/p^s)$ than of this quotient.

Proof. This is more a discussion of the ideas than an actual proof. First, the expansions for logarithm and exponential show that for s large enough,

$$\exp(\mathfrak{g}^{(s)}(\mathcal{O}_{\mathfrak{p}})) = \mathbf{G}^{(s)}(\mathcal{O}_{\mathfrak{p}}).$$

But without Hensel's lemma, we need a better definition of the Haar measure on $\mathbf{G}$. We detour briefly into the general construction of the Haar measure from a top form.

Let K be a p-adic field. A function $K^n \to K$ is analytic if can be written locally as a convergent power series. A K-analytic manifold is defined by an atlas of charts to K^n whose transition functions are K-analytic. The linear algebraic group $\mathbf{G}(k_{\mathfrak{p}})$ is a $k_{\mathfrak{p}}$ analytic manifold.

K-analytic functions are differentiable in the sense that they are locally approximated by linear maps. Once you have derivatives, tangent spaces and differential forms make sense too, since we can define them in terms of how they transform under the transition maps between charts. For example a top form on a K-analytic manifold M is an object that transforms by the rule

$$\varphi^*(f dt_1 \wedge \cdots \wedge dt_n) = f\det\frac{\partial \varphi_i}{\partial x_j} dt_1 \wedge \cdots \wedge dt_n.$$

where $dt_1 \wedge \cdots \wedge dt_n$ stands for the volume form on K^n and f is a K-analytic function. In the real setting, a top form ω determines a measure by integrating against the absolute value of ω. The following proposition shows that this makes sense over a K-analytic manifold M as well.

Proposition 54. *[7] Let U and V be open subsets of K^n and let φ be a bi-analytic isomorphism from U to V. If $f : V \to \mathbb{C}$ is measurable, then*

$$\int_V f d\mu_V = \int_U f(\varphi)\left|\det\left(\frac{\partial \varphi_i}{\partial x_j}(x)\right)\right|_K d\mu_U.$$

Namely, we can define the integral of a top form by pulling it back to a chart in K^n, and this proposition shows that it doesn't matter which chart we pull back to.

Now we return to our linear algebraic group $\mathbf{G}$. Since the group $\mathbf{G}(k_{\mathfrak{p}})$ is a $k_{\mathfrak{p}}$-analytic manifold, top forms on it determine measures. A top form on $\mathfrak{g}(k_{\mathfrak{p}})$ determines a unique left-invariant top form on $\mathbf{G}(k_{\mathfrak{p}})$, and we take the measure corresponding to this form.

Now to compute the volume of $\mathbf{G}^{(s)}(\mathcal{O}_\mathfrak{p}) = \exp(\mathfrak{g}^{(s)}(\mathcal{O}_\mathfrak{p}))$ the proposition shows that we just need to integrate the absolute value of the determinant of the derivative of the exponential map over $\mathfrak{g}^{(s)}(\mathcal{O}_p)$. The following lemma of Poincare and Schur shows that for large enough s, the absolute value of this determinant is one.

Proposition 55. *Let A and X be vectors in $\mathfrak{g}(\mathbb{Q}_p)$ and let B be the linear transformation* $\mathrm{ad}A$ *of* $\mathfrak{g}(\mathbb{Q}_p)$.

$$\left.\frac{d}{dt}\right|_{t=0} \exp(A+tX) = \exp(A)\sum_{j=0}^{\infty} \frac{B^j}{(j+1)!}X$$

Since the top form on $\mathbf{G}$ is left invariant, the Jacobian determinant is just

$$\det\left(\sum_{j=0}^{\infty} \frac{B^j}{(j+1)!}\right).$$

If $A \in \mathfrak{g}^{(s)}(\mathbb{Z}_p)$ for s large enough, then the absolute value of this determinant is one. Therefore, for s large enough, the volume of $\mathbf{G}^{(s)}(\mathcal{O}_\mathfrak{p})$ is the same as the volume of $\mathfrak{g}^{(s)}(\mathcal{O}_p)$, which proves the theorem.

□

Given any semisimple linear algebraic group with a representation, and a top form on its Lie algebra, we have formulas (25) and (26) for the $\mathfrak{p}$-adic volumes at its points of good reduction and bad reduction respectively. If we fix a top form ω on the Lie algebra defined over k, then we can normalize all the $\mathfrak{p}$-adic Haar measures uniformly. Remember, we want to eventually be multiplying the different $\mathfrak{p}$-adic volumes together. We hope that the resulting infinite product converges. We point out that convergence is independent of our choice of the representation and our choice of ω since changing either of these only affects finitely many primes. In particular, whatever the lattice $\mathfrak{g}(\mathcal{O}_k)$ and the volume form, the quantity $\mathrm{vol}_\omega \mathfrak{g}(\mathcal{O}_p)$ will be equal to one for all but finitely many primes. Therefore, the question of convergence is settled affirmatively by the following theorem.

Theorem 56. *([17], Theorem 6.4) For any semisimple linear algebraic group* $\mathbf{G}$ *over a number field k, the product*

$$\prod_{\mathfrak{p}\ \textit{prime}} \frac{|\mathbf{G}(\mathbb{F}_\mathfrak{p})|}{N(\mathfrak{p})^{-\dim \mathbf{G}}}$$

is convergent.

This shows that for any representation and for any volume form ω,

$$\prod_{\mathfrak{p}\ \text{prime}} \mathrm{vol}_\omega(\mathbf{G}(\mathcal{O}_\mathfrak{p}))$$

is convergent. Of course, the value still depends both on the representation and the choice of ω. However, it turns out that this is only because we forgot to include the infinite places. In fact if we look at all places simultaneously, both infinite and finite, we can get a number which depends only on the linear algebraic group $\mathbf{G}$ itself. To see this, we introduce the adeles.

Let k be a number field, and let $\mathbb{P}$ be the set of all places of k, both infinite and finite. Let S be a finite set of places which contains the set $\mathbb{P}^\infty$ of infinite places. If σ is a place of k, denote the corresponding completion of k by k^σ, so if σ is infinite then k^σ is either $\mathbb{R}$ or $\mathbb{C}$ and if $\sigma = \sigma_\mathfrak{p}$ is a finite place, then k^σ is $k_\mathfrak{p}$. Define

$$A_S := \prod_{\sigma \in S} k^\sigma \prod_{\sigma_\mathfrak{p} \notin S} \mathcal{O}_\mathfrak{p}$$

and define the adeles of k to be the colimit

$$A_k := \varinjlim_S A_S$$

When the field is understood, we will also refer to this simply as A. The ring A splits as a product $A^\infty \times A^f$ where

$$A^\infty := \prod_{\sigma \in \mathbb{P}^\infty} k^\sigma$$

are the 'infinite adeles' and

$$A^\sigma := \prod_{\mathfrak{p} \text{ prime}}{}' k_\mathfrak{p}$$

are the 'finite adeles.' Here $\prod'$ is shorthand for the colimit of the products where all but finitely many terms are required to be integral.

Each completion of k comes with an absolute value, which induces a topology. Although the infinite product of the absolute values does not extend to an absolute value on the adeles, we can always take products and colimits of topological spaces. Since the addition and multiplication is continuous in each completion, it is in the adeles as well. Therefore the adeles are a topological ring. Furthermore, since each completion is locally compact, the adeles are locally compact. An observation that we will use often is that a local basis for the topology is given by products of open sets in each completion, where all but finitely many of the open sets are equal to $\mathcal{O}_\mathfrak{p}$.

Another important fact is that the product of the measures on each completion defines a Haar measure on the adeles. The only condition that requires checking is that the product measure is finite on compact sets, and this follows from the normalization that the $\mathfrak{p}$-adic measure of each ring of integer $\mathcal{O}_\mathfrak{p}$ is one.

Since any element of k lies in $\mathcal{O}_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$, the field k embeds into its ring of adeles. With respect to the topology on A_k, the image of k is discrete. Indeed, we have

$$k \cap A_{\mathbb{P}^\infty} \cong \mathcal{O}_k$$

where $A_{\mathbb{P}^\infty}$ is the open set $A^\infty \times \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}$ as defined above. We have already seen that $\mathcal{O}_k$ sits discretely inside $\mathbb{A}^\infty$, so certainly $k \cap A_{\mathbb{P}^\infty}$ is discrete inside $A_{\mathbb{P}^\infty}$. Since k is a subgroup of the adeles, and any one element is isolated, it follows that k is discrete.

In fact, we showed that $\mathcal{O}_k$ was a lattice inside A^∞, the covolume of which was $\sqrt{\Delta_k}$. It's natural now to ask whether k is a lattice in the adeles and if so what its covolume is. We can answer this using the following observation.

Proposition 57. *The projection of k to the finite adeles is dense.*

Proof. An open subset of $k_{\mathfrak{p}}$ is determined by a congruence condition modulo some power of p. Since a basis of the topology of the finite adeles is given by the product of open sets, all but finitely many of which are $\mathcal{O}_p$, this proposition is equivalent to showing that if we specify some finite set of congruence conditions, we can find a rational number that satisfies them. Hence, this proposition is a reformulation of the Chinese remainder theorem. $\square$

Since $A_{\mathbb{P}^\infty}$ is open, this proposition shows that for any adele $a \in A$, k intersects $a + A_{\mathbb{P}^\infty}$ nontrivially. In other words,

$$A = k \cdot A_{\mathbb{P}^\infty}$$

Therefore,

$$k \backslash A = (k \cap A_{\mathbb{P}^\infty}) \backslash A_{\mathbb{P}^\infty} = \mathcal{O}_k \backslash A_{\mathbb{P}^\infty}$$

This maps to $\mathcal{O}_k \backslash A^\infty$ with kernel A^c. Since A^c has volume one, the pushforward of this measure to A^∞ is the standard measure on A^∞. Therefore,

$$\operatorname{vol}(k \backslash A) = \operatorname{vol}(\mathcal{O}_k \backslash A^\infty) = \sqrt{\Delta_k}.$$

Now suppose that $\mathbf{G}$ is a group defined over k. Note that the $\operatorname{Res}_{k/\mathbb{Q}} \mathbf{G}(\mathbb{R}) = \mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) = \mathbf{G}(A^\infty)$. The theorem of Borel and Harish-Chandra (44) showed that $\mathbf{G}(\mathcal{O}_k)$ is a lattice inside of $\mathbf{G}(A^\infty)$. The example of the discriminant above shows that if we want to find a group in which $\mathbf{G}(k)$ is a lattice, we should try $\mathbf{G}(A)$. Since the adeles are locally compact with measure, groups defined over the adeles also have a Haar measure. We would like to say that this Haar measure is just the product of the Haar measures at each place, but we need to check that the product measure is finite on compact sets. In fact, this is true by Theorem (56). This theorem shows that the topology of $\mathbf{G}(\mathbb{A})$ is generated by open sets of finite measure; since any compact set can be covered by finitely many of these, any compact set has finite measure.

The great thing about the Haar measure on $\mathbf{G}(A)$ is that defining it in this way specifies it canonically, *not* up to a multiplicative constant. Indeed, suppose instead of starting with the top form ω on $\mathfrak{g}$, we had started with the top form $t\omega$ for $t \in k$. Then the measure at each place changes by the corresponding absolute value of t. So the measure on $\mathbf{G}(A)$ changes by the product of the absolute values of t with respect to each place. Taking the product over the infinite places, you get what we've been calling $N(t)$. At the finite places, you get by definition

$$\prod_{\mathfrak{p}} N(\mathfrak{p})^{-\nu_{\mathfrak{p}}(t)} = N(\prod_{\mathfrak{p}} \mathfrak{p}^{-\nu_{\mathfrak{p}}(t)}) = N(t^{-1})$$

Therefore, even though the Haar measure at each place changes, their product, the adelic Haar measure, does not depend on the choice of volume form!

Therefore, the quantity

$$\operatorname{vol}(\mathbf{G}(k) \backslash \mathbf{G}(\mathbb{A}))$$

which may a priori be infinite, depends neither on any representation of $\mathbf{G}$ nor on the choice of top form on $\mathfrak{g}$. We saw that if $\mathbf{G}$ is the additive group, then this quantity is exactly $\sqrt{\Delta_k}$. Therefore if we define

$$\tau_k(\mathbf{G}) = \Delta_k^{-\frac{\dim \mathbf{G}}{2}} \operatorname{vol}(\mathbf{G}(k) \backslash \mathbf{G}(\mathbb{A})) \tag{27}$$

then for any field k, τ_k of the additive group is one.

Definition 58. *If* $\mathbf{G}$ *is a semisimple linear algebraic group defined over* k*, then* τ_k *as defined by Equation (27) is the Tamagawa number of* $\mathbf{G}$ *with respect to* k*.*

Remark 59. *The normalization of the Tamagawa number by the discriminant of* k *also gives it the natural property*

$$\tau_k \mathrm{Res}_{L/k}\mathbf{G} = \tau_L \mathbf{G}$$

for any group $\mathbf{G}$ *defined over* L*. Hence it makes sense to define* $\tau(\mathbf{G})$ *without reference to the field.*

We're very close to being able to interpret Equation (21) as the statement that the Tamagawa number of SL_n is one. In order to do that, we need one more major theorem.

Theorem 60. *(Strong approximation, weak version) Let* $\mathbf{G}$ *be a simply connected semisimple linear algebraic group such that* $\mathbf{G}(\mathbb{R})$ *is not compact. Then the projection of* $\mathbf{G}(k)$ *to* $\mathbf{G}(A^f)$ *is dense.*

A fairly elementary proof for SL_n is given in the notes [3]. Note that Proposition (57) is strong approximation for the additive group.

Since SL_n is defined over $\mathbb{Z}$, we have no problem defining $SL_n(A_{\mathbb{P}^\infty})$, which is equal to

$$SL_n(\mathbb{R}) \times \prod_p SL_n(\mathbb{Z}_p)$$

This is an open subgroup of $SL_n(A)$, so strong approximation implies that

$$SL_n(A) = SL_n(k) \cdot SL_n(A_{\mathbb{P}^\infty}).$$

Therefore,

$$\begin{aligned} SL_n(\mathbb{Q})\backslash SL_n(A) &= (SL_n(\mathbb{Q}) \cap SL_n(A_{\mathbb{P}^\infty})\backslash SL_n(A_{\mathbb{P}^\infty}) \\ &= SL_n(\mathbb{Z})\backslash SL_n(A_{\mathbb{P}^\infty}) \end{aligned}$$

Now fixing our standard volume form ω, so that we can talk about the volume of each component, we have

$$\mathrm{vol}(SL_n(\mathbb{Q})\backslash SL_n(A_{\mathbb{Q}})) = \mathrm{vol}_\omega(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})) \cdot \prod_p \mathrm{vol}_\omega SL_n(\mathbb{Z}_p).$$

The discriminant of $\mathbb{Q}$ is one, so is exactly the Tamagawa number. Therefore, after all that work we've finally managed to interpret our magic formula as the statement that the Tamagawa number of SL_n is one. Perhaps this gives no more indication of why it is true, but the next theorem shows that it puts the group SL_n in good company.

Theorem 61. *(Weil conjecture on Tamagawa numbers) If* $\mathbf{G}$ *is a connected, simply connected semisimple group then* $\tau(\mathbf{G}) = 1$*.*

This theorem was proved in many stages by lots of different people; the final case of the proof was settled in 1989. It ultimately involves reducing to the case of Chevalley groups proved by Langlands.

Remark 62. *If* $\mathbf{G}$ *is not simply connected, there is also a fairly simple formula for* $\tau(\mathbf{G})$*, but it includes some cohomological terms that take a minute to define [17]. It's pretty immediate from Borel and Harish-Chandras theorem that* $\tau(\mathbf{G})$ *is always finite. More interestingly, it's always rational.*

2.5 Covolumes of arithmetic Fuchsian and Kleinian groups

In this final section, we combine our study of the arithmetic subgroups of SL_2 with the deep result on Tamagawa numbers in the previous section to give a formula for the covolumes of arithmetic lattices in $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$. A discrete subgroup of $SL_2(\mathbb{R})$ is called Fuchsian and a discrete subgroup of $SL_2(\mathbb{C})$ is called Kleinian. The only difference between the two cases is whether or not our field k has a complex place, and so we'll be able to do both the real case and the complex case at the same time.

We will find the covolume of one arithmetic subgroup in each commensurability class. By our classification of the k-forms of SL_2 (Section 1.4), the commensurability classes of arithmetic groups are in bijection with pairs (k, S) where k is a totally real number field and S is a finite set of places of k of even cardinality containing all but one of the infinite places. In each commensurability class we would like to pick an arithmetic subgroup which will make the calculation the easiest.

In fact, there is a natural choice. Let D be the quaternion algebra ramified at the set S of places of k. An *order* M is an $\mathcal{O}_k$-lattice in D which is also a ring with identity. It's not hard to show that every order in D is contained in a maximal order. We can also define an order of the localization $D_\mathfrak{p}$ to be an $\mathcal{O}_\mathfrak{p}$-lattice which is also a ring with identity, and similarly every order in $D_\mathfrak{p}$ is contained in a maximal order. The following proposition relates these.

Proposition 63. *([10], Corollary 6.2.8) An order M in D is maximal if and only if $M_\mathfrak{p}$ is maximal in $D_\mathfrak{p}$ for each prime $\mathfrak{p}$ in k.*

Now let $\mathbf{G} = SL_1(D)$ be the corresponding algebraic group over k. The order M determines an arithmetic subgroup $\mathbf{G}(\mathcal{O}_k) \subset \mathbf{G}(k)$, namely the set of points that stabilize M. For each prime $\mathfrak{p}$, we also have the group $\mathbf{G}(\mathcal{O}_\mathfrak{p}) \subset \mathbf{G}(k_\mathfrak{p})$ stabilizing $M_\mathfrak{p}$. If an element of $\mathbf{G}(k)$ stabilizes M, then certainly it stabilizes $M_\mathfrak{p}$ for every $\mathfrak{p}$, and the converse is true as well. In other words, if we define

$$\mathbf{G}(A_{\mathbb{P}^\infty}) := \mathbf{G}(A^\infty) \times \prod_{\mathfrak{p}} \mathbf{G}(\mathcal{O}_\mathfrak{p})$$

then $\mathbf{G}(\mathcal{O}_k) = \mathbf{G}(k) \cap \mathbf{G}(A_{\mathbb{P}^\infty})$. Now since $SL_1(D)$ is simply connected, strong approximation holds, so we can mimic the calculation of the Tamagawa number of SL_2:

$$\begin{aligned}
\mathrm{vol}(\mathbf{G}(k)\backslash\mathbf{G}(A)) &= \mathrm{vol}(\mathbf{G}(k)\backslash(\mathbf{G}(k)\cdot\mathbf{G}(A_{\mathbb{P}^\infty}))) \\
&= \mathrm{vol}(\mathbf{G}(\mathcal{O}_k)\backslash\mathbf{G}(A_{\mathbb{P}^\infty})) \\
&= \mathrm{vol}_\omega(\mathbf{G}(\mathcal{O}_k)\backslash\mathbf{G}(A^\infty)) \cdot \prod_{\mathfrak{p}} \mathrm{vol}_\omega(\mathbf{G}(\mathcal{O}_\mathfrak{p}))
\end{aligned}$$

where we've made a choice of the volume form ω on $\mathfrak{g}(k)$.

Now there's a question about the normalization of ω that I haven't quite figured out. It seems natural to choose ω to be the volume form that gives the lattice $\mathfrak{g}(\mathcal{O}_k)$ volume $\sqrt{\Delta_k}$. The Lie algebra $\mathfrak{g}$ is just the purely imaginary quaternions. But I can't quite relate this to the choice of measure that Maclachlan and Reid make in [10]. The important thing is to be able in the end to relate the measure to the standard measure on $SL_2(\mathbb{R})$.

In any case, Maclachlan and Reid calculate the local volumes with respect to their measure, and get the following results:

Proposition 64. *([10], Lemma 7.5.3) Let* $\mathrm{Ram}_f(D)$ *denote the set of finite places at which* D *ramifies. Choose a maximal order in* D *with respect to which we define the groups* $SL_1(D)(\mathcal{O}_\mathfrak{p})$*. Then*

$$\mathrm{vol}_\omega(SL_1(D)(\mathcal{O}_\mathfrak{p})) = \begin{cases} 1 - N(\mathfrak{p})^{-2} & \text{if } \mathfrak{p} \notin \mathrm{Ram}_f(D) \\ (1 - N(\mathfrak{p})^{-2})(N(\mathfrak{p}) - 1)^{-1} & \text{if } \mathfrak{p} \in \mathrm{Ram}_f(D) \end{cases} \tag{28}$$

Also, $\mathbf{G}(A^\infty) = SL_2(\mathbb{R}) \times SU(2) \times \cdot \times SU(2)$, so

$$\mathrm{vol}(\mathbf{G}(\mathcal{O}_k)\backslash\mathbf{G}(A^\infty) = \mathrm{vol}(\mathbf{G}(\mathcal{O}_k)\backslash SL_2(\mathbb{R})) \cdot \mathrm{vol}(SU(2))^n.$$

At any ramified infinite place, they have

$$\mathrm{vol}_\omega(SU(2)) = 4\pi^2 \tag{29}$$

Putting together Equations (28) and (29) with the theorem that $\tau(\mathbf{G}) = 1$ finally yields a formula for the covolume of the arithmetic subgroup.

Theorem 65. *Let* $F = \mathbb{R}$ *or* $\mathbb{C}$*. Let* k *be a number field that has one place* σ_0 *in* F *and exactly* s *other infinite places, all in* $\mathbb{R}$*. Let* D *be a quaternion algebra over* k *which ramifies at all infinite places other than* σ_0*. Let* $M \subset D$ *be a maximal order, and let* $SL_1(D)(\mathcal{O}_k)$ *be the arithmetic subgroup of* $SL_1(D)(k)$ *which preserves* M*. Then*

$$\mathrm{vol}(SL_1(D)(\mathcal{O}_k)\backslash SL_2(F)) = \frac{|\Delta_k|^{3/2}\zeta_k(2)\prod_{\mathfrak{p}\in\mathrm{Ram}_f(D)}(N(\mathfrak{p}) - 1)}{(4\pi^2)^s}$$

This is with respect to the standard volume form on $SL_2(F)$. We can now translate this formula into a formula for the volume of the corresponding locally symmetric space. We'll do the real case. We recall that the standard volume form on $SL_2(\mathbb{R})$ was that which gave the lattice $SL_2(\mathbb{Z})$ covolume one. A basis for this lattice is given by

$$E := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad F := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad G := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The subalgebra $\mathrm{Lie}(SO(2))$ is the span on E, so F and G give a basis for the tangent space to the symmetric space $SL_2(\mathbb{R})/SO(2) \cong H^2$. We want to compute the volume of $\Gamma\backslash H^2$ with respect to the volume form of the metric of curvature -1, which determines a form ω_{H^2}. We need to divide by the volume of $SO(2)$ with respect to the form $\omega_{SO(2)}$ such that

$$1 = \omega_{SL_2}(E, F, G) = \omega_{SO(2)}(E)\omega_{H^2}(F, G)$$

So we need to explicitly write down ω_{H^2} as an element of

$$T_pH^2 = \mathrm{Lie}(SL_2(\mathbb{R}))/\mathrm{Lie}(SO(2)).$$

To figure out $\omega_{H^2}(F, G)$, we could use the standard formulas for the curvature of a Riemannian submersion, but instead we'll start with the perhaps more familiar fact that the hyperbolic metric on H^2 is given by

$$g_{\text{hyp}} = \frac{dx^2 + dy^2}{y^2}$$

where $x = \text{Re}(z)$ and $y = \text{Im}(z)$. Recall also that $SL_2(\mathbb{R})$ acts on H^2 by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}$$

and the stabilizer of the point i is $SO(2)$. Let ρ be the pushforward map from Lie$(SL_2(\mathbb{R})$ to $T_i H^2$. Then

$$\rho(F) = \frac{d}{dt}|_{t=0}\exp(tF) \cdot i = \frac{d}{dt}\frac{i + t}{1} = 1 \tag{30}$$

$$\rho(G) = \frac{d}{dt}|_{t=0}\exp(tG) \cdot i = \frac{d}{dt}\frac{e^t i}{e^{-t}} = 2i \tag{31}$$

Therefore, $\omega_{H^2}(F, G) = 2$, so $\omega_{SO(2)}(E) = 1/2$. The volume of $SO(2)$ with respect to this volume form is π.

It follows that to translate the volume in Theorem (65) into hyperbolic volume we need to divide by π assuming the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is in the arithmetic subgroup (otherwise we divide by $\pi/2$).

In the case of $SL_2(\mathbb{C})$, the corresponding symmetric space $SL_2(\mathbb{C})/SU(2)$ is hyperbolic 3-space. We can compute this volume the same way, using $\mathbb{Z}[i]$. This is only a little harder.

The last chapter of [10] summarizes some theorems that can be deduced from this formula. Here's one with some number theory significance that I'm still curious about. We remarked at the beginning that for H^2, in contrast with H^3, the Gauss-Bonnet theorem lets us translate statements about volume into statements about Euler characteristic. The problem is that the Euler form is identically zero on H^3.

Recall the Gauss Bonnet theorem for a closed surface Σ:

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma k$$

Our locally symmetric spaces are not always closed surfaces, but this isn't a problem. If Σ is a hyperbolic surface with cusps, then truncating each cusp with a curve that is nearly geodesic and taking the limit as that curve goes to infinity shows that Gauss-Bonnet still holds. In fact, Harder showed that this is more generally

Theorem 66. *[6] Let G be a semisimple Lie group, and let X be the corresponding symmetric space. Let ω_X be the Euler form of X. If Γ is an arithmetic subgroup of G, then*

$$\int_{\Gamma\backslash X} \omega_X = \chi\Gamma$$

In particular, this holds even if $\Gamma\backslash X$ is noncompact.

So one corollary of our volume formula is that if k is a totally real field then the following quantity is always rational:

$$\frac{\Delta_k^{\frac{1}{2}}\zeta_k(2)}{(4\pi^2)^{[k:\mathbb{Q}]}} \tag{32}$$

There is an interesting coincidence that I'm curious how to explain. The functional equation for the zeta function of a number field says the following. Define

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2); \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

and

$$\Lambda_k(s) = |\Delta_k|^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_k(s).$$

Then

$$\Lambda_k(s) = \Lambda_k(1-s).$$

Applying the functional equation to the quantity (32) shows that this is equivalent to the statement that $\zeta(-1)$ is rational. I don't know why the normalization of the Euler class should involve the same irrational part as the functional equation.

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